$PSU(1,1|2)$ Lie superalgebra homology

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Contents

1 Lie superalgebra homology for singleton

1.1 Commutation relations

The bosonic parts satisfy

$$\begin{aligned} &[J^+, J^-] = 2J^3 \qquad &[J^3, J^\pm] = \pm J^\pm \\ &[\tilde{J}^+, \tilde{J}^-] = 2\tilde{J}^3 \qquad &[\tilde{J}^3, \tilde{J}^\pm] = \pm \tilde{J}^\pm \end{aligned} \tag{1}$$

Compare to [1] equations (3.?) with

$$\begin{aligned} K_1^0 &= J^3 \\ K_2^0 &= \tilde{J}^3 \\ K_1^\pm &= J^\pm \\ K_2^\pm &= \tilde{J}^\pm \end{aligned} \tag{2}$$

Then there are the four fermionic raising and four fermionic lowering operators ($\alpha = 1, 2$)

$$S^\pm_{1\alpha} \qquad\qquad S^\pm_{2\alpha} \tag{3}$$

which raise/lower the eigenvalues of K_1^0, K_2^0 by $\pm\frac{1}{2}$.

Read off the commutation relations from the zero-mode part of the affine algebra copied from [1]

$$[K^0_{1,m}, K^\pm_{1,n}] \;=\; \pm\, K^\pm_{1,m+n} \qquad\qquad [K^0_{2,m}, K^\pm_{2,n}] \;=\; \pm\, K^\pm_{2,m+n} \tag{4}$$

$$[K^0_{1,m}, S^\pm_{1\alpha,n}] \;=\; \pm\frac{1}{2}\, S^\pm_{1\alpha,m+n} \qquad\qquad [K^0_{1,m}, S^\pm_{2\alpha,n}] \;=\; \pm\frac{1}{2}\, S^\pm_{2\alpha,m+n} \tag{5}$$

$$[K^0_{2,m}, S^\pm_{1\alpha,n}] \;=\; \pm\frac{1}{2}\, S^\pm_{1\alpha,m+n} \qquad\qquad [K^0_{2,m}, S^\pm_{2\alpha,n}] \;=\; \mp\frac{1}{2}\, S^\pm_{2\alpha,m+n} \tag{6}$$

$$\{S^\pm_{1\alpha,m}, S^\pm_{2\beta,n}\} \;=\; \mp 2\epsilon_{\alpha\beta}\, K^\pm_{1,m+n} \qquad\qquad \{S^\pm_{1\alpha,m}, S^\mp_{2\beta,n}\} \;=\; \pm 2\epsilon_{\alpha\beta}\, K^\pm_{2,m+n} \tag{7}$$

$$[K^\pm_{1,m}, S^\mp_{1\alpha,n}] \;=\; \pm S^\pm_{2\alpha,m+n} \qquad\qquad [K^\pm_{1,m}, S^\mp_{2\alpha,n}] \;=\; \mp S^\pm_{1\alpha,m+n} \tag{8}$$

$$[K^\pm_{2,m}, S^\mp_{1\alpha,n}] \;=\; \pm S^\mp_{2\alpha,m+n} \qquad\qquad [K^\pm_{2,m}, S^\pm_{2\alpha,n}] \;=\; \mp S^\pm_{1\alpha,m+n} \quad . \tag{9}$$

In addition, there are six relations involving the level k of the current algebra. These read as follows,

$$[K^0_{1,m}, K^0_{1,n}] \;=\; -\frac{k}{2}\, m\, \delta_{m+n,0} \qquad\qquad [K^0_{2,m}, K^0_{2,n}] \;=\; \frac{k}{2}\, m\, \delta_{m+n,0} \tag{10}$$

$$[K^+_{1,m}, K^-_{1,n}] \;=\; 2K^0_{1,m+n} - mk\, \delta_{m+n,0} \qquad\qquad [K^+_{2,m}, K^-_{2,n}] \;=\; 2K^0_{2,m+n} + mk\, \delta_{m+n,0} \tag{11}$$

$$\{S^+_{1\alpha,m}, S^-_{1\beta,n}\} \;=\; 2\epsilon_{\alpha\beta}\big(K^0_{1,m+n} - K^0_{2,m+n}\big) - 2\, mk\, \epsilon_{\alpha\beta}\, \delta_{m+n,0} \tag{12}$$

$$\{S^+_{2\alpha,m}, S^-_{2\beta,n}\} \;=\; 2\epsilon_{\alpha\beta}\big(K^0_{1,m+n} + K^0_{2,m+n}\big) - 2\, mk\, \epsilon_{\alpha\beta}\, \delta_{m+n,0} \quad . \tag{13}$$

1.2 Quadratic twistor free field representation

For $psu(1,1|2)$ non-affine singleton have bosonic and fermionic oscillators for $a, b, i, j = 1, 2$

$$\begin{aligned} [\alpha^\dagger_a, \alpha^b] &= \delta^b_a \\ \{c^\dagger_i, c^j\} &= \delta^j_i \end{aligned} \tag{14}$$

Introduce bosonic generators

$$\begin{aligned} &J^+ = \alpha^\dagger_2 \alpha^1 \qquad J^- = \alpha^\dagger_1 \alpha^2 \qquad J^3 = \frac{1}{2}(\alpha^\dagger_1\alpha^1 - \alpha^\dagger_2\alpha^2) \\ &\tilde{J}^+ = -c^\dagger_2 c^1 \qquad \tilde{J}^- = -c^\dagger_1 c^2 \qquad \tilde{J}^3 = \frac{1}{2}(c^\dagger_2 c^2 - c^\dagger_1 c^1) \\ &C = \alpha^\dagger_1\alpha^1 + \alpha^\dagger_2\alpha^2 - c^\dagger_1 c^1 - c^\dagger_2 c^2 \\ &B = \alpha^\dagger_1\alpha^1 + \alpha^\dagger_2\alpha^2 \end{aligned} \tag{15}$$

C commutes with everything, B has non-trivial commutators with the fermionic generators.

The fermionic generators (deduced by $[C, S] = 0$ and the commutation relations)

$$\begin{aligned}
S^+_{11} &= c^\dagger_2 \alpha^1 + \alpha^\dagger_2 c^1 \\
S^-_{11} &= c^\dagger_1 \alpha^2 + \alpha^\dagger_1 c^2 \\
S^+_{12} &= -c^\dagger_2 \alpha^1 + \alpha^\dagger_2 c^1 \\
S^-_{12} &= -c^\dagger_1 \alpha^2 + \alpha^\dagger_1 c^2 \\
S^+_{21} &= c^\dagger_1 \alpha^1 - \alpha^\dagger_2 c^2 \\
S^-_{21} &= -c^\dagger_2 \alpha^2 + \alpha^\dagger_1 c^1 \\
S^+_{22} &= -c^\dagger_1 \alpha^1 - \alpha^\dagger_2 c^2 \\
S^-_{22} &= c^\dagger_2 \alpha^2 + \alpha^\dagger_1 c^1
\end{aligned} \tag{16}$$

Using

$$\{c^\dagger_b \alpha^a, \alpha^\dagger_c c^d\} = \alpha^\dagger_c \alpha^a \delta^d_b - c^\dagger_b c^d \delta^a_c \tag{17}$$

we can check for example

$$\begin{aligned}
\{S^+_{11}, S^-_{11}\} &= -\{S^+_{12}, S^-_{12}\} = C = 0 \\
\{S^+_{11}, S^-_{12}\} &= -\{S^+_{12}, S^-_{11}\} = 2J^3 - 2\tilde{J}^3 \\
\{S^+_{21}, S^-_{21}\} &= -\{S^+_{22}, S^-_{22}\} = C = 0 \\
\{S^+_{21}, S^-_{22}\} &= -\{S^+_{22}, S^-_{21}\} = 2J^3 + 2\tilde{J}^3
\end{aligned} \tag{18}$$

1.3 Matter singleton states

Vacuum $|1\rangle$ defined by

$$\alpha^\dagger_1 |1\rangle = \alpha^2 |1\rangle = c^\dagger_1 |1\rangle = c^2 |1\rangle = 0 \tag{19}$$

so that $C |1\rangle = 0$.

It is annihilated by the following generators:

$$\left(J^-, \tilde{J}^-, S^-_{1\alpha}, S^-_{2\alpha}, S^+_{2\alpha}\right) |1\rangle \tag{20}$$

This makes it half-BPS.

Acting with the remaining generators gives us the $psu(1,1|2)$ singleton representation, which coincides with all the states we can build with $C = 0$.

oscillator	GT operator	charges $x^C y^{2J^3} z^{2\tilde{J}^3} b^B$
$(J^+)^p \|1\rangle$	$\partial^p_{1\dot{2}} Y$	$x^0 y^{2p+1} z^{-1} b$
$(J^+)^p \tilde{J}^+ \|1\rangle$	$\partial^p_{1\dot{2}} Z$	$x^0 y^{2p+1} z b$
$(J^+)^p \frac{1}{2}(S^+_{11} - S^+_{12}) \|1\rangle = (J^+)^p c^\dagger_2 \alpha^1 \|1\rangle$	$\partial^p_{1\dot{2}} \lambda_{41}$	$x^0 y^{2p+2} b^2$
$(J^+)^p \frac{1}{2}(S^+_{11} + S^+_{12}) \|1\rangle = (J^+)^p \alpha^\dagger_2 c^1 \|1\rangle$	$\partial^p_{1\dot{2}} \bar{\lambda}^1_{\dot{2}}$	$x^0 y^{2p+2} b^0$

Table 1: Singleton states with $C = 0$, $p \geq 0$.

The character is then

$$\frac{b}{1 - y^2} \left(y z^{-1} + y z + y^2 b + y^2 b^{-1}\right) \tag{21}$$

This is the character of the atypical discrete series representation of $psu(1,1|2)$

$$[0]_+ = (-\frac{1}{2}, \frac{1}{2}) \oplus 2(-1, 0) \tag{22}$$

in equation (2.28) of [1].

Could also have chosen the ground state to be a fermion λ instead of the scalar Y. It makes no difference to spectrum.

1.3.1 Action of algebra on singleton states

The only subtle thing here is that

$$S_{11}^{+}S_{12}^{+}|1\rangle = -2\tilde{J}^{+}J^{+}|1\rangle \tag{23}$$

This can be checked from the oscillators or the action of the generators on the singleton from Bianchi and co. [2] discussed below.

On $(J^{+})^{p}|1\rangle$ the action of the algebra elements are:

$$\tilde{J}^{-}, S_{1\alpha}^{-}, S_{2\alpha}^{+} : 0 \tag{24}$$
$$J^{+}, \tilde{J}^{+}, S_{1\alpha}^{+} : \text{ multiplication} \tag{25}$$
$$J^{-} : -p^{2}(J^{+})^{p-1}|1\rangle \tag{26}$$
$$S_{2\alpha}^{-} : pS_{1\alpha}^{+}(J^{+})^{p-1}|1\rangle \tag{27}$$

On $\tilde{J}^{+}(J^{+})^{p}|1\rangle$ the action of the algebra elements are:

$$\tilde{J}^{+}, S_{1\alpha}^{+}, S_{2\alpha}^{-} : 0 \tag{28}$$
$$J^{+} : \text{ multiplication} \tag{29}$$
$$J^{-} : -p^{2}\tilde{J}^{+}(J^{+})^{p-1}|1\rangle \tag{30}$$
$$\tilde{J}^{-} : (J^{+})^{p}|1\rangle \tag{31}$$
$$S_{1\alpha}^{-} : -pS_{1\alpha}^{+}(J^{+})^{p-1}|1\rangle \tag{32}$$
$$S_{2\alpha}^{+} : S_{1\alpha}^{+}(J^{+})^{p}|1\rangle \tag{33}$$

On $S_{11}^{+}(J^{+})^{p}|1\rangle$ the action of the algebra elements are:

$$\tilde{J}^{-}, \tilde{J}^{+}, S_{11}^{+}, S_{11}^{-}, S_{21}^{-}, S_{21}^{+} : 0 \tag{34}$$
$$J^{+} : \text{ multiplication} \tag{35}$$
$$J^{-} : -p(p+1)S_{11}^{+}(J^{+})^{p-1}|1\rangle \tag{36}$$
$$S_{12}^{+} : 2\tilde{J}^{+}(J^{+})^{p+1}|1\rangle \tag{37}$$
$$S_{12}^{-} : 2(p+1)(J^{+})^{p}|1\rangle \tag{38}$$
$$S_{22}^{+} : -2(J^{+})^{p+1}|1\rangle \tag{39}$$
$$S_{22}^{-} : 2(p+1)\tilde{J}^{+}(J^{+})^{p}|1\rangle \tag{40}$$

On $S_{12}^{+}(J^{+})^{p}|1\rangle$ the action of the algebra elements are:

$$\tilde{J}^{-}, \tilde{J}^{+}, S_{12}^{+}, S_{12}^{-}, S_{22}^{-}, S_{22}^{+} : 0 \tag{41}$$
$$J^{+} : \text{ multiplication} \tag{42}$$
$$J^{-} : -p(p+1)S_{12}^{+}(J^{+})^{p-1}|1\rangle \tag{43}$$
$$S_{11}^{+} : -2\tilde{J}^{+}(J^{+})^{p+1}|1\rangle \tag{44}$$
$$S_{11}^{-} : -2(p+1)(J^{+})^{p}|1\rangle \tag{45}$$
$$S_{21}^{+} : 2(J^{+})^{p+1}|1\rangle \tag{46}$$
$$S_{21}^{-} : -2(p+1)\tilde{J}^{+}(J^{+})^{p}|1\rangle \tag{47}$$

1.3.2 Comparison to Bianchi et al. conventions

Comparing to [2], fermion generators are identified with those of $PSU(2,2|4)$: $Q_{1}^{i}, S_{j}^{1}, \bar{Q}_{j2}, \bar{S}^{i2}$ for $i,j = 2,3$. $Z = \varphi_{34}, Y = \varphi_{42}$.

$$[Q_{1}^{2}, Y] = -2\lambda_{41} \tag{48}$$
$$[\bar{Q}_{32}, Y] = \bar{\lambda}_{2}^{1} \tag{49}$$
$$\{Q_{1}^{2}, [\bar{Q}_{32}, Y]\} = \{Q_{1}^{2}, \bar{\lambda}_{2}^{1}\} = -2i\partial_{12}Z \tag{50}$$
$$[Q_{1}^{2}, Z] = 0 \tag{51}$$
$$[\bar{Q}_{32}, Z] = 0 \tag{52}$$

1.4 Ghosts and ghost ground state

$$
\begin{aligned}
\{b^3, c^3\} &= 1 \\
\{b^\pm, c^\mp\} &= 2 \\
[\beta^\pm_{1\alpha}, \gamma^\mp_{1\beta}] &= \epsilon_{\alpha\beta} \\
[\beta^\pm_{2\alpha}, \gamma^\mp_{2\beta}] &= \epsilon_{\alpha\beta}
\end{aligned}
\tag{53}
$$

***I've guessed metric for bosonic ghosts.

Choose ghost vacuum to be

$$
c^{3,\pm}|0\rangle = \tilde{c}^{3,\pm}|0\rangle = \gamma^\pm_{1\alpha}|0\rangle = \gamma^\pm_{2\alpha}|0\rangle = 0 \tag{54}
$$

With this choice of vacuum we are doing BRST *homology* rather than *cohomology*, see Section 4.

1.5 BRST operator

BRST operator is then

$$
\begin{aligned}
Q =& c^3 J^3 + \tfrac{1}{2} c^+ J^- + \tfrac{1}{2} c^- J^+ + \tilde{c}^3 \tilde{J}^3 + \tfrac{1}{2} \tilde{c}^+ \tilde{J}^- + \tfrac{1}{2} \tilde{c}^- \tilde{J}^+ \\
&+ \epsilon^{\alpha\beta} \gamma^\mp_{1\alpha} S^\pm_{1\beta} + \epsilon^{\alpha\beta} \gamma^\mp_{2\alpha} S^\pm_{2\beta} \\
&+ \tfrac{1}{2}(c^+ c^- b^3 + c^- c^3 b^+ + c^3 c^+ b^-) \\
&+ \tfrac{1}{2}(\tilde{c}^+ \tilde{c}^- \tilde{b}^3 + \tilde{c}^- \tilde{c}^3 \tilde{b}^+ + \tilde{c}^3 \tilde{c}^+ \tilde{b}^-) \\
&+ c\gamma\beta + \gamma\gamma b + \tilde{c}\gamma\beta + \gamma\gamma\tilde{b}
\end{aligned}
\tag{55}
$$

***Not quite sure about metric normalisation for quadratic part γS for fermions, cf. (2.2) of [1].

All must be normal-ordered (i.e. put c, γ ghosts to the right).

The remaining 48 cubic terms are:

$$\begin{aligned}
&-2\gamma^-_{11}\gamma^+_{12}b^3+2\gamma^-_{11}\gamma^+_{12}\tilde{b}^3\\
&+2\gamma^-_{12}\gamma^+_{11}b^3-2\gamma^-_{12}\gamma^+_{11}\tilde{b}^3\\
&+2\gamma^-_{22}\gamma^+_{21}b^3+2\gamma^-_{22}\gamma^+_{21}\tilde{b}^3\\
&-2\gamma^-_{21}\gamma^+_{22}b^3-2\gamma^-_{21}\gamma^+_{22}\tilde{b}^3\\
&-2\gamma^-_{12}\gamma^-_{21}b^++2\gamma^-_{11}\gamma^-_{22}b^+\\
&+2\gamma^+_{12}\gamma^+_{21}b^--2\gamma^+_{11}\gamma^+_{22}b^-\\
&+2\gamma^-_{12}\gamma^+_{21}\tilde{b}^+-2\gamma^-_{11}\gamma^+_{22}\tilde{b}^+\\
&-2\gamma^+_{12}\gamma^-_{21}\tilde{b}^-+2\gamma^+_{11}\gamma^-_{22}\tilde{b}^-\\
&+\tfrac{1}{2}c^3\gamma^-_{11}\beta^+_{12}-\tfrac{1}{2}c^3\gamma^+_{11}\beta^-_{12}\\
&-\tfrac{1}{2}c^3\gamma^-_{12}\beta^+_{11}+\tfrac{1}{2}c^3\gamma^+_{12}\beta^-_{11}\\
&-\tfrac{1}{2}c^3\gamma^+_{21}\beta^-_{22}+\tfrac{1}{2}c^3\gamma^-_{21}\beta^+_{22}\\
&+\tfrac{1}{2}c^3\gamma^+_{22}\beta^-_{21}-\tfrac{1}{2}c^3\gamma^-_{22}\beta^+_{21}\\
&+\tfrac{1}{2}c^-\gamma^+_{11}\beta^+_{22}-\tfrac{1}{2}c^-\gamma^+_{12}\beta^+_{21}\\
&-\tfrac{1}{2}c^-\gamma^+_{21}\beta^+_{12}+\tfrac{1}{2}c^-\gamma^+_{22}\beta^+_{11}\\
&-\tfrac{1}{2}c^+\gamma^-_{11}\beta^-_{22}+\tfrac{1}{2}c^+\gamma^-_{12}\beta^-_{21}\\
&+\tfrac{1}{2}c^+\gamma^-_{21}\beta^-_{12}-\tfrac{1}{2}c^+\gamma^-_{22}\beta^-_{11}\\
&+\tfrac{1}{2}\tilde{c}^3\gamma^-_{11}\beta^+_{12}-\tfrac{1}{2}\tilde{c}^3\gamma^+_{11}\beta^-_{12}\\
&-\tfrac{1}{2}\tilde{c}^3\gamma^-_{12}\beta^+_{11}+\tfrac{1}{2}\tilde{c}^3\gamma^+_{12}\beta^-_{11}\\
&+\tfrac{1}{2}\tilde{c}^3\gamma^+_{21}\beta^-_{22}-\tfrac{1}{2}\tilde{c}^3\gamma^-_{21}\beta^+_{22}\\
&-\tfrac{1}{2}\tilde{c}^3\gamma^+_{22}\beta^-_{21}+\tfrac{1}{2}\tilde{c}^3\gamma^-_{22}\beta^+_{21}\\
&+\tfrac{1}{2}\tilde{c}^-\gamma^+_{11}\beta^-_{22}-\tfrac{1}{2}\tilde{c}^-\gamma^+_{12}\beta^-_{21}\\
&+\tfrac{1}{2}\tilde{c}^-\gamma^-_{22}\beta^+_{11}-\tfrac{1}{2}\tilde{c}^-\gamma^-_{21}\beta^+_{12}\\
&+\tfrac{1}{2}\tilde{c}^+\gamma^-_{12}\beta^+_{21}-\tfrac{1}{2}\tilde{c}^+\gamma^-_{11}\beta^+_{22}\\
&-\tfrac{1}{2}\tilde{c}^+\gamma^+_{22}\beta^-_{11}+\tfrac{1}{2}\tilde{c}^+\gamma^+_{21}\beta^-_{12}
\end{aligned}\tag{56}$$

The Cartan ghost currents are then

$$\begin{aligned}
J^{3,gh}=\{Q^{gh},b^3\}=&\tfrac{1}{2}b^+c^--\tfrac{1}{2}b^-c^++\tfrac{1}{2}\beta^+_{12}\gamma^-_{11}-\tfrac{1}{2}\beta^-_{12}\gamma^+_{11}\\
&-\tfrac{1}{2}\beta^+_{11}\gamma^-_{12}+\tfrac{1}{2}\beta^-_{11}\gamma^+_{12}-\tfrac{1}{2}\beta^-_{22}\gamma^+_{21}+\tfrac{1}{2}\beta^+_{22}\gamma^-_{21}+\tfrac{1}{2}\beta^-_{21}\gamma^+_{22}-\tfrac{1}{2}\beta^+_{21}\gamma^-_{22}\\
\tilde{J}^{3,gh}=\{Q^{gh},\tilde{b}^3\}=&\tfrac{1}{2}\tilde{b}^+\tilde{c}^--\tfrac{1}{2}\tilde{b}^-\tilde{c}^++\tfrac{1}{2}\beta^+_{12}\gamma^-_{11}-\tfrac{1}{2}\beta^-_{12}\gamma^+_{11}\\
&-\tfrac{1}{2}\beta^+_{11}\gamma^-_{12}+\tfrac{1}{2}\beta^-_{11}\gamma^+_{12}+\tfrac{1}{2}\beta^-_{22}\gamma^+_{21}-\tfrac{1}{2}\beta^+_{22}\gamma^-_{21}-\tfrac{1}{2}\beta^-_{21}\gamma^+_{22}+\tfrac{1}{2}\beta^+_{21}\gamma^-_{22}
\end{aligned}\tag{57}$$

Under these currents the ghosts are charged exactly as their Lie algebra counterparts.

1.6 Homology

Physical state must be uncharged under the two total Cartans $J^{3,total}=\{Q,b^3\}$ and $\tilde{J}^{3,total}=\{Q,\tilde{b}^3\}$.

To cancel non-compact charge of matter part, need to use bosonic β ghosts.

The second index of the $S^{\pm}_{a\alpha}$ or $\beta^{\pm}_{a\alpha}$ is unchanged under $PSU(2|2)$ and is related to an additional $SL(2,\mathbb{R})$ or $SU(2)$ automorphism of the algebra (very specific to $PSU(2|2)$). The structure is unchanged by Q, so we can split the state spaces up into those with a new quantity called **value**, which is defined by

$$\text{value} := \#\text{ of 1's} - \#\text{ 2's}\tag{58}$$

antighost number value	1	2	3	4	5	6	7	8	$2n$	$2n+1$
$2n+1$									0	1
$2n-1$									0	1
9								0	1	1
7						0	1	0	1	1
5				0	1	0	1	1	1	1
3		0	1	0	1	1	1	1	1	1
1	1	0	1	1	1	1	1	1	1	1
-1	1	0	1	1	1	1	1	1	1	1
-3		0	1	0	1	1	1	1	1	1
-5				0	1	0	1	1	1	1
-7						0	1	0	1	1
-9								0	1	1
$-2n+1$									0	1
$-2n-1$									0	1
total	2	0	4	2	6	4	8	6	$2n-2$	$2n+2$

Table 2: Number of states in homology for singleton.

antighost number value	1	2	3	4	5	6	7	8
9								10-10
7						6-6	51-50	202-202
5				3-3	28-27	114-114	327-326	766-765
3		1-1	12-11	51-51	154-153	382-381	810-809	1557-1556
1	3-2	13-13	44-43	127-126	299-298	633-632	1211-1210	2160-2159

Table 3: Number of closed states minus exact states for singleton.

antighost number value	1	2	3	4	5	6	7	8
9								10
7						6	57	252
5				3	31	141	441	1092
3		1	13	62	205	535	1191	2366
1	3	15	57	170	425	931	1843	3370

Table 4: Total number of states with eligible charges for singleton.

1.7 Homology examples

1.7.1 Ghost number -1

At this number there are 6 states (for $\alpha = 1, 2$) with the requisite charges and they are all Q-closed

$$
\begin{aligned}
QS^+_{1\alpha}b^- \left|1\right\rangle =&0\\
Q\tilde{J}^+\beta^-_{1\alpha} \left|1\right\rangle =&0\\
Q\beta^-_{2\alpha} \left|1\right\rangle =&0
\end{aligned}
\tag{59}
$$

4 linear combinations of these are Q-exact (see next section), e.g. for $\alpha = 1$

$$
\begin{aligned}
-\beta^-_{21}\left|1\right\rangle + S^+_{11}b^-\left|1\right\rangle\\
S^+_{11}b^-\left|1\right\rangle + \tilde{J}^+\beta^-_{11}\left|1\right\rangle
\end{aligned}
\tag{60}
$$

We could take the two representatives of the homology to be $\beta^-_{2\alpha}\left|1\right\rangle$.

1.7.2 Ghost number -2

Total of 32 possible states. Leave out 12 states which are just b^3 or $\tilde{b}^3$ times states at ghost number -1. We know these are Q-closed.

For value 1 with $\alpha = 1$ overall there are 9 states

$$\begin{aligned}
QJ^+b^-\beta^-_{21}\left|1\right\rangle =& -\beta^-_{21}\left|1\right\rangle + S^+_{11}b^-\left|1\right\rangle \\
Qb^-\beta^+_{11}\left|1\right\rangle =& -\beta^-_{21}\left|1\right\rangle + S^+_{11}b^-\left|1\right\rangle \\
Q\tilde{J}^+b^-\beta^+_{21}\left|1\right\rangle =& S^+_{11}b^-\left|1\right\rangle + \tilde{J}^+\beta^-_{11}\left|1\right\rangle \\
Q\tilde{b}^+\beta^-_{11}\left|1\right\rangle =& \beta^-_{21}\left|1\right\rangle + \tilde{J}^+\beta^-_{11}\left|1\right\rangle \\
Q\beta^-_{11}\beta^-_{22}S^+_{11}\left|1\right\rangle =& 2S^+_{11}b^-\left|1\right\rangle + 2\tilde{J}^+\beta^-_{11}\left|1\right\rangle \\
Q\beta^-_{12}\beta^-_{21}S^+_{11}\left|1\right\rangle =& 2\beta^-_{21}\left|1\right\rangle - 2S^+_{11}b^-\left|1\right\rangle \\
Q\beta^-_{11}\beta^-_{21}S^+_{12}\left|1\right\rangle =& -2\beta^-_{21}\left|1\right\rangle - 2\tilde{J}^+\beta^-_{11}\left|1\right\rangle \\
QJ^+\tilde{J}^+b^-\beta^-_{11}\left|1\right\rangle =& -S^+_{11}b^-\left|1\right\rangle - \tilde{J}^+\beta^-_{11}\left|1\right\rangle \\
Q\tilde{J}^+\tilde{b}^-\beta^-_{21}\left|1\right\rangle =& \beta^-_{21}\left|1\right\rangle + \tilde{J}^+\beta^-_{11}\left|1\right\rangle
\end{aligned} \tag{61}$$

Similarly for value -1 with $\alpha = 2$.

For values 3 and -3 there are one each

$$\begin{aligned}
Q\beta^-_{11}\beta^-_{21}S^+_{11}\left|1\right\rangle =& 0 \\
Q\beta^-_{12}\beta^-_{22}S^+_{12}\left|1\right\rangle =& 0
\end{aligned} \tag{62}$$

$32 = 12 + 9 + 9 + 2$

1.7.3 Representatives for odd ghost number

For odd antighost number $2n+1$ there is 1 state in homology for maximum value $2n+1$. This can be represented with either

$$\tilde{J}^+(\beta^+_{11}\beta^-_{11})^n\beta^-_{11} \tag{63}$$

or

$$(\beta^+_{21}\beta^-_{21})^n\beta^-_{21} \tag{64}$$

For the former one can move down with the $SU(2)$ generator. The state

$$n\tilde{J}^+(\beta^+_{11})^{n-1}b^+_{12}(\beta^-_{11})^{n+1} + (n+1)\tilde{J}^+(\beta^+_{11}\beta^-_{11})^n\beta^-_{12} \tag{65}$$

has value $2n-1$. It is closed and not exact. Checked up to $n = 3$.

For the latter one can also move down with the $SU(2)$ generator. The state

$$n(\beta^+_{21})^{n-1}b^+_{22}(\beta^-_{21})^{n+1} + (n+1)(\beta^+_{21}\beta^-_{21})^n\beta^-_{22} \tag{66}$$

has value $2n-1$. It is closed and not exact. Checked up to $n = 3$.

1.7.4 Representatives for even ghost number

For even antighost number $2n+4$ there is 1 state in homology for value $2n+1$. This can be represented simply with

$$(\beta^+_{11}\beta^-_{11} + \beta^+_{21}\beta^-_{21})^n S^+_{11}b^-\tilde{b}^+\tilde{b}^3\tilde{b}^-\left|1\right\rangle \tag{67}$$

For up to $n = 2$ have checked that it is closed and not exact.

Can we move down with even generators?

1.8 Potential $psu(1,1|2)$ action on states

It looks like there is some J^+-like action that takes us from the top-valued odd and even homology to their counterparts two antighost numbers higher.

Search through all the combinations of fields with antighost number 2 and value 2 that are uncharged under the total Cartans. If we exclude things of the form $\beta\beta\beta\gamma$, then they have the general form $J\beta\beta$ or $Sb\beta$. Some examples:

$$\begin{aligned} J^+\beta^-_{11}\beta^-_{21} \\ \beta^+_{11}\beta^-_{11} \\ \beta^+_{21}\beta^-_{21} \\ S^+_{21}b^3\beta^-_{21} \\ J^+S^+_{21}b^-\beta^-_{21} \end{aligned} \tag{68}$$

There are $4 \times 5/2 = 10$ of the form $J\beta\beta$ (excluding multiplying by J^3 or $\tilde{J}^3$) and $6 \times 4 = 24$ of the form $Sb\beta$.

Even if some linear combination of these doesn't commute with Q, it could commute up to the total Cartans which annihilate all relevant states, see below.

There also seems to be a fermionic operator that takes us from antighost number one to antighost number four, e.g. $\tilde{b}^+\tilde{b}^-\tilde{b}^3$ takes us from $S^+_{11}b^-|1\rangle$ to $S^+_{11}b^-\tilde{b}^+\tilde{b}^3\tilde{b}^-|1\rangle$.

1.8.1 General principle of commuting action

Suppose we have a state $|\psi\rangle$ in homology

$$Q|\psi\rangle = 0, \qquad \nexists\, |\phi\rangle \text{ such that } |\psi\rangle = Q|\phi\rangle \tag{69}$$

Suppose there is also an operator $\hat{J}^+$ that acts on $|\psi\rangle$ and commutes with Q on this state

$$[Q, \hat{J}^+]\,|\psi\rangle = 0 \tag{70}$$

Suppose in addition that there is an operator $\hat{J}^-$ such that

$$\hat{J}^-\hat{J}^+|\psi\rangle \propto |\psi\rangle \tag{71}$$

and such that on all states with vanishing total Cartan charges

$$[Q, \hat{J}^-]\,|\phi\rangle = 0 \qquad \forall\phi \tag{72}$$

[Probably this implies that $[Q, \hat{J}^-] = \sum_i c_i J_i^{3,total}$ for the total Cartan charges.]

Then the state $\hat{J}^+|\psi\rangle$ is also in Q homology.

Proof: it is Q-closed by (70). Prove it is not Q-exact by contradiction. If it is Q-exact, then so is $|\psi\rangle$, since if

$$\hat{J}^+|\psi\rangle = Q|\phi\rangle \tag{73}$$

then

$$|\psi\rangle \propto \hat{J}^-Q|\phi\rangle \tag{74}$$

by (71). Then by (72)

$$|\psi\rangle \propto Q\hat{J}^-|\phi\rangle \tag{75}$$

Contradiction! So $\hat{J}^+|\psi\rangle$ is not Q-exact. QED.

2 Extension to other atypical discrete series representations

In our $\mathcal{N}=4$-friendly notation (see `bmodel-u112-*.dvi`) for the bosonic decomposition of the atypicals for $n \in \mathbb{N}$, $n \geq 2$

$$\left[\frac{n-1}{2}\right]_+ = (-\tfrac{n}{2}, \tfrac{n}{2}) \oplus 2(-\tfrac{n}{2} - \tfrac{1}{2}, \tfrac{n}{2} - \tfrac{1}{2}) \oplus (-\tfrac{n}{2} - 1, \tfrac{n}{2} - 1) \tag{76}$$

This has a slightly different HWS to reference [1]. $n = 1$ reduces to singleton, which is slightly simpler.

2.1 States

Assume have LWS defined by

$$\begin{aligned}
\tilde{J}^3 |n\rangle &= -\tfrac{n}{2} |n\rangle \\
J^3 |n\rangle &= \tfrac{n}{2} |n\rangle \\
\tilde{J}^- |n\rangle &= 0 \\
J^- |n\rangle &= 0 \\
(\tilde{J}^+)^{n+1} |n\rangle &= 0 \\
(S^-_{1\alpha}, S^-_{2\alpha}, S^+_{2\alpha}) |n\rangle &= 0
\end{aligned} \tag{77}$$

So states are

$$(\tilde{J}^+)^p (J^+)^q |n\rangle \tag{78}$$

for $p = 0, \dots n$ and $q = 0, 1, 2 \dots$ and charges $y^{2J^3} z^{2\tilde{J}^3} = y^{2q+n} z^{2p-n}$.

$$S^+_{1\alpha} (\tilde{J}^+)^p (J^+)^q |n\rangle \tag{79}$$

for $\alpha = 1, 2$, $p = 0, \dots n-1$ and $q = 0, 1, 2 \dots$ and charges $y^{2q+n+1} z^{2p-n+1}$.

$$S^+_{11} S^+_{12} (\tilde{J}^+)^p (J^+)^q |n\rangle \tag{80}$$

for $p = 0, \dots n-2$ and $q = 0, 1, 2 \dots$ and charges $y^{2q+n+2} z^{2p-n+2}$.

Ground state $|n\rangle$ is like half-BPS state $\mathrm{tr}(Y^n)$ for $\mathcal{N} = 4$.

2.2 Action of algebra on states

Non-trivial relation:

$$S^+_{11} S^+_{12} (\tilde{J}^+)^{n-1} |n\rangle = -\frac{2}{n} (\tilde{J}^+)^n J^+ |n\rangle \tag{81}$$

Can check this either by using realisation of rep n as symmetric product of singleton or by applying $\tilde{J}^-$ to both sides.

2.3 Homology for $n = 2$

antighost number value	1	2	3	4	5	6	7	8	$2n-1$	$2n$
$2n$									0	1
$2n-2$									0	1
8							0	1	1	1
6					0	1	0	1	1	1
4			0	1	0	1	1	1	1	1
2		1	0	1	1	1	1	?	1	1
0	0	1	0	1	1	1	1	?	1	1
total	0	3	0	5	3	7	5	?	$2n-3$	$2n+1$

Table 5: Number of states in homology for $n = 2$.

2.3.1 Representatives of homology

For antighost number 2 and value 2 any of the five eligible states work, e.g.

$$\beta^-_{21} \beta^-_{21} |2\rangle \tag{82}$$

For ghost 2 and value 0 the following works

$$\beta_{22}^{-}\beta_{21}^{-}\left|2\right\rangle \tag{83}$$

For ghost $2n$ and value $2n$ the following works

$$(\beta_{11}^{+}\beta_{11}^{-})^{n-1}\beta_{22}^{-}\beta_{21}^{-}\left|2\right\rangle \tag{84}$$

2.4 Homology for higher n

See tables.

antighost number value	1	2	3	4	5	6	7	8	9	$2n$	$2n+1$
$2n+1$										0	1
$2n-1$										0	1
9								0	1	1	1
7						0	1	0	1	1	1
5				0	1	0	1	1	?	1	1
3			1	0	1	1	1	?	?	1	1
1		0	1	0	1	1	1	?	?	1	1
total	0	0	4	0	6	4	8	?	?	$2n-2$	$2n+2$

Table 6: Number of states in homology for $n = 3$.

antighost number value	1	2	3	4	5	6	7	8	9	10	$2n-1$	$2n$
$2n$											0	1
$2n-2$											0	1
10									0	1	1	1
8							0	1	0	?	1	1
6					0	1	0	1	1	?	1	1
4				1	0	1	1	1	?	?	1	1
2			0	1	0	1	1	?	?	?	1	1
0			0	1	0	1	1	?	?	?	1	1
total	0	0	0	5	0	7	5	?	?	?	$2n-3$	$2n+1$

Table 7: Number of states in homology for $n = 4$.

antighost number value	1	2	3	4	5	6	7	8	9	10	$2n$	$2n+1$
$2n+1$											0	1
$2n-1$											0	1
11										0	1	1
9								0	1	?	1	1
7						0	1	0	1	?	1	1
5					1	0	1	1	?	?	1	1
3				0	1	0	1	?	?	?	1	1
1				0	1	0	1	?	?	?	1	1
total	0	0	0	0	6	0	8	?	?	?	$2n-2$	$2n+2$

Table 8: Number of states in homology for $n = 5$.

2.5 General features of homology

Homology starts at antighost number n, where $n+1$ states appear in a representation of $su(2)$. At antighost number $n+1$ there is nothing. At antighost number $n+2$ there are $n+3$ states.

Then pattern establishes itself.

Note that for $n+4$ and above the pattern for n and $n+2$ agree.

Horizontally always have $(0)|1|0|1|1|1\cdots$.

3 Extension to finite dimensional atypicals

3.1 Results in the literature

In `http://webdoc.sub.gwdg.de/ebook/serien/e/mpi_mathematik/2003/39.ps` "SuperLie and problems (to be) solved with it" by Pavel Grozman and Dimitry Leites they describe their Mathematica program SuperLie for calculating Lie superalgebra (co)homology and some (limited) calculations they can do with it.

On page 21 there are the following statements for the trivial representation $\mathbb{C}$ (they abbreviate $H^*(g;\mathbb{K})$ to $H^*(g)$ where $\mathbb{K}$ is their field)

$$H^*(gl(m|n)) \cong H^*(gl(\max(m,n))) \tag{85}$$

$$H^*(psl(n|n)) \cong H^*(sl(n)) \otimes \mathbb{K}[h_2] \text{ for } n > 2$$

$$H^*(psl(2|2)) \cong \mathbb{C}[x_2, y_2, h_2, \xi_3]/(x_2 y_2 - h_2^2) \tag{86}$$

where h_2 represents the extension $sl(n|n) \to psl(n|n)$. x_2, y_2 and h_2 form some kind of $sl(2)$, a bit like ours? Again you can see here that $psl(2|2)$ is a bit exceptional in the $psl(n|n)$ series.

Original calculation was in [FL]. This missed cocycle ξ_3. The answer was rectified at our request by A. Shapovalov with the help of SuperLie, see also [KK1].

[FL]: Fuks Leites D.B. FUKS and D.A. LEITES, Cohomology of Lie superalgebras, Comptes Rendus de l'Académie Bulgare des Sciences, tome 37, n. 12 (1984).

[KK1] Kornyak, `http://arxiv.org/abs/math.SC/9906046`, `http://arxiv.org/abs/math.SC/0002210`

See also:

"Cohomology of Lie superalgebras and their generalizations" by M. Scheunert and R. Zhang, Journal of Mathematical Physics 39 5024-5061 (1998) `http://arxiv.org/abs/q-alg/9701037`.

"Cohomology of Lie superalgebras $sl_{m|n}$ and $osp_{2|2n}$" by Yucai Su and R.B. Zhang (2004) `http://arxiv.org/abs/math/0402419`.

Finitude de l'homologie de certains modules de dimension finie sur une super algèbre de Lie `http://aif.cedram.org/aif-bin/item?id=AIF_1997__47_2_531_0` C. Gruson, Ann. Inst. Fourier, Grenoble 47, 531 (1997) also has the result (85) for $gl(m|n)$ and trivial $\mathbb{C}$ rep in THeorem 5.3.

3.2 Trivial representation

Assume have matter state $|0\rangle$ killed by all generators. Only get interesting potential states by building up invariant things with ghosts.

antighost number value	0	1	2	3	4	5	6	7	8	9	$2n$	$2n+1$
$2n$											1	0
$2n-2$											1	1
8									1	0	1	1
6							1	0	1	?	1	1
4					1	0	1	1	1	?	1	1
2			1	0	1	1	1	1	1	?	1	1
0	1	0	1	1	1	1	1	1	1	?	1	1
total	1	0	3	1	5	3	7	5	9	?	$2n+1$	$2n-1$

Table 9: Number of states in homology for trivial representation.

Representative for homology at ghost 2 and value 2

$$\left(\beta_{11}^{+}\beta_{11}^{-} + \beta_{21}^{+}\beta_{21}^{-}\right)|0\rangle \tag{87}$$

Representative for homology at ghost $2n$ and value $2n$

$$\left(\beta_{11}^{+}\beta_{11}^{-} + \beta_{21}^{+}\beta_{21}^{-}\right)^{n}|0\rangle \tag{88}$$

Representative for homology at ghost 2 and value 0

$$\left(\beta_{11}^{+}\beta_{12}^{-} + \beta_{21}^{+}\beta_{22}^{-} + \beta_{12}^{+}\beta_{11}^{-} + \beta_{22}^{+}\beta_{21}^{-}\right)|0\rangle \tag{89}$$

Or first pair or second pair on their own.

Representative for homology at ghost 3 and value 0

$$\left(b^{+}b^{-}b^{3}\right)|0\rangle \tag{90}$$

$\tilde{b}^{+}, \tilde{b}^{-}, \tilde{b}^{3}|0\rangle$ also works.

Representative for homology at ghost 5 and value 2

$$\left(b^{+}b^{-}b^{3}\right)\left(\beta_{11}^{+}\beta_{11}^{-} + \beta_{21}^{+}\beta_{21}^{-}\right)|0\rangle \tag{91}$$

A single one of the terms also works. An $SU(2)$ descendant of this works at value 0.

Similarly for ghost $2n+1$ and value $2n-3$.

Q: Can we translate these representatives to the literature result in (86)?

3.3 Adjoint

14-dimensional adjoint. It is contained in the Kac module $[0,0] : [0] \to [\frac{1}{2}] \to [0]$. $[0]$ is the 1-dim trivial and $[\frac{1}{2}]$ is the 14-dim adjoint.

Need to know full action of algebra on all states in order to work out homology.

From [1]:

Finite dimensional Kac modules of $psl(2|2)$ are labelled by pairs $[j_1, j_2]$ for $j_i = 0, \frac{1}{2}, 1, \ldots$. A Kac module $[j_1, j_2]$ is irreducible whenever $j_1 \neq j_2$. In case $j_1 = j_2$, however, Kac modules turn out to be indecomposable composites of smaller irreducible building blocks (short multiplets).

[For Kac module take HWS and act with all fermionic generators.]

Adjoint has character

$$2\chi_{\frac{1}{2}}(z_1)\chi_{\frac{1}{2}}(z_2) + \chi_1(z_1) + \chi_1(z_2) \tag{92}$$

Represent the states as $|A\rangle$ where $B|A\rangle = f_{AB}^{C}|C\rangle$.

There are eligible states with correct charges at any antighost number, e.g.

$$\begin{aligned} &(\beta_{1\alpha_i}^{+}\beta_{1\beta_j}^{-})^{n} b^{-} \left|J^{+}\right\rangle \\ &(\beta_{1\alpha_i}^{+}\beta_{1\beta_j}^{-})^{n} \beta_{11}^{+} \left|S_{11}^{-}\right\rangle \end{aligned} \tag{93}$$

All states at antighost number one are closed.

3.4 Other finite-dimensional atypicals

No calculations were done for the finite-dimensional atypicals beyond the trivial and the adjoint.

The problem is that for the (co)homology the computer program has to know the action of each generator on each state in the representation. This was easy for the trivial and the adjoint, but harder for the singleton (see Section 1.3.1) and so tedious for the higher discrete series irreps that I didn't bother to type it up (although a hand-written version is available on request; you can also see it in the computer code). It's easy enough to get the action correct for the bosonic generators, but very fiddly to get it right for the fermionic guys, which take you between the bosonic reps. There are signs and coefficients which make things ugly.

However we will sketch here how it should work and compare it to the adjoint.

antighost number value	0	1	2	3	4	5	6	7	8	$2n-1$	$2n$
$2n$										1	0
$2n-2$										2	1
8								1	0	2	2
6						1	0	2	1	2	2
4				1	0	2	1	2	2	2	2
2		1	0	2	1	2	2	2	?	2	2
0	0	1	0	2	1	2	2	?	?	2	2
total	0	3	0	8	3	12	8	16?	12?	$4n$	$4n-4$

Table 10: Number of states in homology for adjoint representation.

For $j > 0$

$$\chi_{[j]}(z_1, z_2) = 2\chi_j(z_1)\chi_j(z_2) - \chi_{j+\frac{1}{2}}(z_1)\chi_{j-\frac{1}{2}}(z_2) - \chi_{j-\frac{1}{2}}(z_1)\chi_{j+\frac{1}{2}}(z_2) \tag{94}$$

In general we have states

$$|j_1, m_1; j_2, m_2; a\rangle \tag{95}$$

where a is the multiplicity of 2 in the first bosonic irrep.

For the adjoint $j = \frac{1}{2}$ the HWS are

$$\begin{aligned} \left|\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2}; 1\right\rangle &= \left|S^+_{11}\right\rangle \\ \left|\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2}; 2\right\rangle &= \left|S^+_{12}\right\rangle \end{aligned} \tag{96}$$

The HWS is killed by 5 fermionic generators and 2 bosonic ones

$$(S^+_{1\alpha}, S^-_{11}, S^+_{21}, S^-_{21}, J^+, \tilde{J}^+) \left|\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2}; 1\right\rangle = 0 \tag{97}$$

The Cartans give self-explanatory results; then the remaining actions are

$$\begin{aligned} J^- \left|\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2}; 1\right\rangle &= \left|\tfrac{1}{2}, -\tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2}; 1\right\rangle = -\left|S^-_{21}\right\rangle \\ \tilde{J}^- \left|\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2}; 1\right\rangle &= \left|\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{1}{2}, -\tfrac{1}{2}; 1\right\rangle = -\left|S^+_{21}\right\rangle \\ S^+_{22} \left|\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2}; 1\right\rangle &= |1, 1; 0, 0\rangle = -2\left|J^+\right\rangle \\ S^-_{22} \left|\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2}; 1\right\rangle &= |0, 0; 1, 1\rangle = 2\left|\tilde{J}^+\right\rangle \\ S^-_{12} \left|\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2}; 1\right\rangle &= -|0, 0; 1, 0\rangle - |1, 0; 0, 0\rangle = -2\left|\tilde{J}^3\right\rangle + 2\left|J^3\right\rangle \end{aligned} \tag{98}$$

Continue until all actions on all states are elucidated.

Some of these actions will generalise to the general j case, but there may be j-dependent factors that need to be determined explicitly. This was true for the general discrete series atypicals, but in that case we had an explicit realisation of these states from $\mathcal{N} = 4$.

Perhaps it is possible to get the general j states by some sort of tensor product of the adjoint?

4 Lie superalgebra cohomology versus homology

The ghost vacuum choice that we made in equation (54) following Tanaka for $gl(1|1)$, with the vacuum killed by all c and γ ghosts, means that we are doing *homology* with states built up with antighosts.

For the *cohomology* one would use a ghost vacuum killed by all antighosts

$$b^{3,\pm}|0\rangle = \tilde{b}^{3,\pm}|0\rangle = \beta^{\pm}_{1\alpha}|0\rangle = \beta^{\pm}_{2\alpha}|0\rangle = 0 \tag{99}$$

and then build up states with c and γ so that they have positive ghost number.

For finite-dimensional algebras and representations, the homology and cohomology groups are known to be isomorphic, see page 19 of `http://webdoc.sub.gwdg.de/ebook/serien/e/mpi_mathematik/2003/39.ps`.

Calculations with computer programs confirmed this.

Also the infinite-dimensional atypical discrete series representations were checked up to around ghost number. They were also found to have the same size of cohomology groups at each ghost number and value as the homology.

5 Lift to affine case

From [1]:

The conformal dimension for a $PSU(1,1|2)$ WZW state is

$$h_{[(\pm,j_1),j_2]} = \frac{1}{k}\left(-j_1(j_1+1) + j_2(j_2+1)\right) \tag{100}$$

Atypical or BPS representations have zero conformal dimensions, from the formula

$$h_{[(\pm,-j-1),j]} = \frac{1}{k}\left(-(j+1)j + j(j+1)\right) = 0 \tag{101}$$

So the G/G calculation for the 1-spectrally flowed sector should reduce to the Lie super-algebra calculation with J sector as the singleton derived from the twistor free field rep and vacuum in I sector.

6 Relation to $AdS_3 \times S^3$

See for example recent paper [3] that covers some of the rep. theory too.

7 Comparison with $GL(1|1)$

Bosonic ghosts (i.e. corresponding to fermionic group Lie algebra elements) are needed to pad out states so that their total Cartan charges are zero.

Have to understand how the Lie superalgebra homology relates to Vladimir's calculation. Is Vladimir two copies, right- and left-moving, or is it one copy that splits in two? Remember there that we reduce to $PSU(1|1)$.

There should be at least one state from the cohomological reduction to $PSU(1|1)$ - but what group action do we use for the reduction?

References

[1] G. Gotz, T. Quella and V. Schomerus, "The WZNW model on $PSU(1,1|2)$," JHEP **0703** (2007) 003 [arXiv:hep-th/0610070].

[2] M. Bianchi, F. A. Dolan, P. J. Heslop and H. Osborn, "N=4 superconformal characters and partition functions," Nucl. Phys. B **767** (2007) 163 [arXiv:hep-th/0609179].

[3] M. R. Gaberdiel and S. Gerigk, "The massless string spectrum on $AdS_3 \times S^3$ from the supergroup," arXiv:1107.2660 [hep-th].