From $c = 1$ to Hurwitz to localization on CP1 string moduli space

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1 Motivation

The multi-trace two-point function of half-BPS operators in $\mathcal{N} = 4$ or the tachyon amplitudes for the $c = 1, R = 1$ string can be written as sums over holomorphic maps from the worldsheet to $\mathbb{CP}^1$. The partitions specifying two branch points come from the operator content while the partition for the third branch point is constrained by the genus.

The sum over holomorphic maps smells like the A-twisted topological string, so it is very natural to expect a map to the Gromov-Witten of $\mathbb{CP}^1$. But how are these very particular branched covers selected? One idea is to use the Atiyah-Bott localization theory to force the integral over the moduli space $\overline{\mathcal{M}}_{g,n}(\mathbb{CP}^1, d)$ of degree d holomorphic maps from an n-pointed genus g curve to $\mathbb{CP}^1$ to localize on the fixed points of a torus action. The natural torus action to choose is the one induced from rotations of the sphere around its central axis. The two fixed points then provide a basis for the equivariant cohomology, the gravitational descendants of which should map to the negative/positive momentum tachyons. An encouraging sign that this is on the correct track is that the fixed points in $\overline{\mathcal{M}}_{g,n}(\mathbb{CP}^1, d)$ are classified by bipartite graphs of genus g (see Kontsevich

[5]), just like the Z complex matrix model of [12]. There are some complications arising from degenerate worldsheets at the boundary of the compactified moduli space $\overline{\mathcal{M}}_{g,n}$, which we hope should be resolved by the better understanding of the completed cycle technology outlined below.

In the meantime it has been easier to make a map to the relative Gromov-Witten theory, where we allow boundaries on the worldsheet that wrap around points on $\mathbb{CP}^1$. This theory is the open-open dual of the Z model.

2 From Casimirs to completed cycles

The $\frac{1}{N}$ expansion of free $U(N)$ gauge theories is governed by the dimensions of $U(N)$ representations R (cf. equation (8)). These dimensions can be written as the exponential of a sum of $C_k(R)$'s, each the leading term of the k'th Casimir,

$$\dim_N R = \frac{d_R}{n!} N^n \exp\left\{ \sum_{k=1}^{\infty} (-1)^k \frac{C_{k+1}(R)}{kN^k} \right\} \tag{1}$$

Following Rudd [13] equation (2.5) the $C_k(R)$'s can be written as a sum over the lengths R_i of the $len(R)$ rows of R

$$C_k(R) = \sum_{i=1}^{len(R)} \sum_{j=1}^{R_i} (j-i)^{k-1} \tag{2}$$

They may also be written as the characters of a sum of cut-and-join operators, e.g. $C_2(R) = \frac{\chi_R(\Sigma_{[2]})}{d_R}$, coming from the exponention of Ω_n, see Section A.

In the paper by Okounkov and Pandharipande "Gromov-Witten theory, Hurwitz theory and completed cycles" [1] the shifted symmetric power sums $\mathbf{p}_k(R)$, defined in equation (0.14), play a very important role

$$\mathbf{p}_k(R) = \sum_{i=1}^{len(R)} \left[(R_i - i + \tfrac{1}{2})^k - (-i + \tfrac{1}{2})^k \right] + (1 - 2^{-k})\zeta(-k) \tag{3}$$

These are just the symmetric group characters of the completed cycles, see Section B.

The $\mathbf{p}_k(R)$ correspond to insertions of the gravitational descendant $\tau_k(\omega)$ of the Kähler class ω in the relative Gromov-Witten theory of $\mathbb{CP}^1$, see Section 3.

It will also be convenient to define the shifted power sum without the final $\zeta(-k)$ term (this R-independent term should be related to a factor $\mathrm{vol}(U(N))$ from the vacuum partition function of $c = 1$ or $\mathbb{CP}^1$, see Section C)

$$OP_k(R) = \mathbf{p}_k(R) - (1 - 2^{-k})\zeta(-k) = \sum_{i=1}^{len(R)} \left[(R_i - i + \tfrac{1}{2})^k - (-i + \tfrac{1}{2})^k \right] \tag{4}$$

Question: How are the $\mathbf{p}_k(R)$ or $OP_k(R)$ related to the $C_k(R)$? If we can answer this question we can relate $\mathcal{N} = 4/c = 1$ amplitudes to the Gromov-Witten theory of $\mathbb{CP}^1$.

Answer: They are related by

$$\boxed{\mathbf{p}_k(R) = \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \frac{1}{2^{2i}} \binom{k}{2i+1} C_{k-2i}(R) + \left(1 - \frac{1}{2^k}\right) \zeta(-k)} \tag{5}$$

$\lfloor\frac{k-1}{2}\rfloor$ means the integer floor of $\frac{k-1}{2}$. Removing the $\zeta(-k)$ term we get

$$\boxed{OP_k(R) = \sum_{i=0}^{\lfloor\frac{k-1}{2}\rfloor} \frac{1}{2^{2i}} \binom{k}{2i+1} C_{k-2i}(R)} \tag{6}$$

For example:

$$\begin{aligned}
OP_2(R) &= 2C_2(R) \\
OP_3(R) &= 3C_3(R) + \frac{1}{4}C_1(R) \\
OP_4(R) &= 4C_4(R) + C_2(R) \\
OP_5(R) &= 5C_5(R) + \frac{5}{2}C_3(R) + \frac{1}{16}C_1(R) \\
OP_6(R) &= 6C_6(R) + 5C_4(R) + \frac{3}{8}C_2(R) \\
OP_7(R) &= 7C_7(R) + \frac{35}{4}C_5(R) + \frac{21}{16}C_3(R) + \frac{1}{64}C_1(R)
\end{aligned} \tag{7}$$

The point: The OP_k are basically the C_k coming from the exponentiation of $\dim_N(R)$, along with lower C_{k-2i} corrections. We would like to interpret these as the contributions of degenerate singular worldsheets from the boundary of the moduli space of Riemann surfaces.

3 From $\mathcal{N} = 4/c = 1$ to relative Gromov-Witten theory of $\mathbb{CP}^1$

In "Complex matrix model duality" [12] the generating function for $\mathcal{N} = 4$ half-BPS two-point functions was written using Kontsevich variables $t_k = \frac{1}{k}\operatorname{tr}(A^k)$ (notation: switch R for λ)

$$\begin{aligned}
\mathcal{Z}(\{t\},\{\bar{t}\}) &= \int [dZ]^{\mathbb{C}}_{N\times N} \ \exp\left(-\operatorname{tr}(ZZ^\dagger) + \sum_{k=1}^{\infty} t_k \operatorname{tr}(Z^k) + \sum_{k=1}^{\infty} \bar{t}_k \operatorname{tr}(Z^{\dagger k}) \right) \\
&= \sum_d \sum_{\lambda\vdash d} \frac{d!\chi_\lambda(A)\chi_\lambda(B)\dim_N\lambda}{d_\lambda} \qquad (8) \\
&= \sum_d \sum_{\mu,\nu,\lambda\vdash d} \operatorname{tr}(\alpha_\mu A)\operatorname{tr}(\alpha_\nu B) N^d \frac{|[\mu]||[\nu]|}{(d!)^2} \chi_\lambda(\alpha_\mu)\chi_\lambda(\alpha_\nu) \exp\left\{ \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{kN^k} C_{k+1}(\lambda) \right\} \qquad (9)
\end{aligned}$$

In [1] Okounkov and Pandharipande calculate the stationary Gromov-Witten invariants of $\mathbb{CP}^1$ relative to $0,\infty \in \mathbb{CP}^1$. Stationary means that we only consider gravitational descendants $\tau_k(\omega)$ of the Kähler form ω, but not of the point class $\tau_k(1)$. Relative means that we have open string worldsheets with boundaries wrapping around 0 and ∞ in the target. The winding profiles are given by partitions μ and ν. The result is (equation (3.1) of [1])

$$\left\langle \mu, \prod_{i=1}^{n} \tau_{k_i}(\omega), \nu \right\rangle^{\bullet\mathbb{CP}^1} = \frac{|[\mu]||[\nu]|}{d!^2} \sum_{|\lambda|\vdash d} \chi_\lambda(\alpha_\mu)\chi_\lambda(\alpha_\nu) \prod_{i=1}^{n} \frac{\mathbf{p}_{k_i+1}(\lambda)}{(k_i+1)!} \tag{10}$$

The τ-function from (4.1) of [1] is given by summing with coefficients

$$\tau_{\mathbb{CP}^1}(\{x\},\{t\},\{\bar{t}\}) = \sum_{|\mu|=|\nu|} tr(\alpha_\mu A)\,\mathrm{tr}(\alpha_\nu B)\left\langle \mu, \exp\left\{\sum_{k=0}^{\infty} x_i \tau_k(\omega)\right\}, \nu \right\rangle^{\bullet \mathbb{CP}^1}$$
$$= \sum_d \sum_{\mu,\nu,\lambda\vdash d} \mathrm{tr}(\alpha_\mu A)\,\mathrm{tr}(\alpha_\nu B)\frac{|[\mu]||[\nu]|}{d!^2}\chi_\lambda(\alpha_\mu)\chi_\lambda(\alpha_\nu)\exp\left\{\sum_{k=0}^{\infty} x_k \frac{\mathbf{p}_{k+1}(\lambda)}{(k+1)!}\right\} \quad (11)$$

With $x_k = \frac{1}{N^k}$ this is very reminiscent of the $\mathcal{N}=4$ generating function (9), except that we've substituted $C_k(\lambda)$ for $\mathbf{p}_k(\lambda)$, which we know from (5) are closely related.

Question: Can we reinterpret the relation (5) between $C_k(\lambda)$ and $\mathbf{p}_k(\lambda)$ as some sort of degenerate worldsheet contribution from the boundary of the moduli space, i.e. where an entire handle is mapped to a single point on the target space? This might make sense seeing that the OP_k include the lower C_{k-2i} as "corrections" to C_k.

Minor comment: Given that the worldsheet of the $\mathbb{CP}^1$ relative GW theory has holes described by μ and ν and vertices for the operator insertions $\tau_k(\omega)$, the worldsheet is graph-dual to the original Feynman graph of $\mathcal{N}=4$. So this theory is the open-open dual of the half-BPS sector of $\mathcal{N}=4$.

4 Hope: relation to equivariant Gromov-Witten theory of $\mathbb{CP}^1$

It would be nicer to relate the $c=1$ amplitudes to the equivariant GW theory of $\mathbb{CP}^1$ described in OP's second paper [2], which is related in a certain limit to the relative GW theory described above. This is still not fleshed out. Some of the problems may be resolved by a reinterpretation of the above results.

In Kontsevich "Enumeration Of Rational Curves Via Torus Actions" [5] it was described how to compute the Gromov-Witten theory of $\mathbb{CP}^1$ with equivariant localization using the $U(1)$ action on $\mathbb{CP}^1$ that rotates the sphere around its equator (leaving the poles fixed).

Consider the moduli space $\overline{\mathcal{M}}_{g,n}(\mathbb{CP}^1,d)$ of degree d stable holomorphic maps from a (possibly nodal) worldsheet of genus g with n marked points to $\mathbb{CP}^1$. Nodal means there may be some degenerate worldsheets where a cycle has pinched and stable means that all connected domains have finite-dimensional automorphism groups preserving the marked points (e.g. exclude spheres with fewer than three marked points).

The $U(1)$ torus action on $\mathbb{CP}^1$ induces a torus action on $\overline{\mathcal{M}}_{g,n}(\mathbb{CP}^1,d)$, whose fixed points are labelled by bipartite graphs. The integration over $\overline{\mathcal{M}}_{g,n}(\mathbb{CP}^1,d)$ we do for GW theory then reduces by the Atiyah-Bott localization theorem to a sum over these fixed points, which are labelled by bipartite graphs, with certain coefficients. The bicolouration of the vertices of the graphs correspond to points on the worldsheet which map to the fixed points on $\mathbb{CP}^1$, i.e. the south and north poles.

[If we map $\overline{\mathcal{M}}_{g,n}(\mathbb{CP}^1,d)$ to $\overline{\mathcal{M}}_{g,n}$ then these discrete fixed points should be exactly those corresponding to algebraic Riemann surfaces.]

One problem with mapping the above description to the Z model of [12] is that the vertices of the graphs are also allowed to represent whole regions of the worldsheet, including handles, which map to a fixed point in the target space.

Question: Are these points in the moduli space the same as the degenerate nodal worldsheets in $\overline{\mathcal{M}}_{g,n}(\mathbb{CP}^1,d)$?

Question: Is incorporating these points the difference between C_k and $\mathbf{p}_k$, which by equation (5), includes corrections in C_{k-2i} to C_k?

Comment: In [9] and earlier references it was pointed out that the coefficient of $\frac{1}{N^{2g}}$ in the expansion of the Z model correlation functions count graphs with the prescribed vertices where g is the *minimum* genus on which the graph can be drawn. But if you were to consider all possible graphs with the prescribed vertices that could be drawn on a surface of genus g then you would have to include lower genus ones too. For example on the torus you also draw planar graphs that ignore the handle. If we include these diagrams too (which would be natural if the string theory theory correlation function has this combinatorial origin) do they correspond to worldsheets at the boundary of moduli space? Is this related to the C_k versus $\mathbf{p}_k$ issue?

4.1 Some equations

The equivariant Poincaré duals of the fixed points $0, \infty \in \mathbb{CP}^1$ form a basis of the localized equivariant cohomology of $\mathbb{CP}^1$

$$\mathbf{0}, \infty \in H^2_{U(1)}(\mathbb{CP}^1, \mathbb{Q}) \tag{12}$$

We want to show that the generating function of the gravitational descendants of these operators is related to the Z model partition function. From equation (3.11) of [2] we have

$$\begin{aligned}\tau(\{t\}, \{\bar{t}\}, u) &= \sum_g \sum_{d\geq 0} u^{2g-2} q^d \left\langle \exp\left(\sum_{i\geq 0} t_i \tau_i(\mathbf{0}) + \bar{t}_i \tau_i(\infty)\right)\right\rangle \\ &= \left\langle e^{\sum t_i \mathbf{A}_i} e^{\alpha_1} \left(\frac{q}{u^2}\right)^H e^{\alpha_{-1}} e^{\sum \bar{t}_i \mathbf{A}^*_i}\right\rangle\end{aligned} \tag{13}$$

The latter is a vacuum expectation in the semi-infinite wedge space $\wedge^{\frac{\infty}{2}} V$.

Challenge: Tranlate this into Hurwitz numbers so we can compare it to the matrix model results.

A Casimirs from cut-and-join operators

From [9]

$$\Omega_n = \exp\left(\frac{1}{N}\Sigma_{[2]} - \frac{1}{2N^2}\left[\binom{n}{2} + \Sigma_{[3]}\right] + \frac{1}{3N^3}\left[(2n-3)\Sigma_{[2]} + \Sigma_{[4]}\right] + \mathcal{O}\left(\frac{1}{N^4}\right)\right) \tag{14}$$

and $\dim_R = \frac{N^n}{n!}\chi_R(\Omega_n)$ so that we get

$$\exp\left\{\sum_{k=1}^{\infty} (-1)^{k+1} \frac{C_{k+1}(R)}{kN^k}\right\} = \frac{1}{d_R}\chi_R(\Omega_n) \tag{15}$$

Using $\frac{\chi_R(\Sigma_\lambda\Sigma_\mu)}{d_R} = \frac{\chi_R(\Sigma_\lambda)}{d_R}\frac{\chi_R(\Sigma_\mu)}{d_R}$ we thus find

$$
\begin{aligned}
&[C_1(R) = n] \\
&C_2(R) = \frac{\chi_R(\Sigma_{[2]})}{d_R} \\
&C_3(R) = \frac{\chi_R(\Sigma_{[3]})}{d_R} + \binom{n}{2} \\
&C_4(R) = \frac{\chi_R(\Sigma_{[4]})}{d_R} + (2n-3)\frac{\chi_R(\Sigma_{[2]})}{d_R} \\
&C_5(R) = \frac{\chi_R(\Sigma_{[5]})}{d_R} + 4\frac{\chi_R(\Sigma_{[2,2]})}{d_R} + (3n-4)\frac{\chi_R(\Sigma_{[3]})}{d_R} + \frac{n(n-1)(4n-5)}{6}
\end{aligned}
\tag{16}
$$

The coefficients $f_k(\sigma)$ for $\sigma \in S_n$ defined by

$$C_k(R) = \sum_{\sigma \in S_n} f_k(\sigma)\frac{\chi_R(\sigma)}{d_R} \tag{17}$$

are class functions. They can be determined spectroscopically using the orthogonality of the characters from the formula (2) for the $C_k(R)$

$$f_k(\tau) = \frac{1}{n!}\sum_{R \vdash n} d_R \chi_R(\tau) C_k(R) \tag{18}$$

B Connection to completed cycles

Following OP [1] equation (0.9) write for $|\eta| \le |\lambda|$ where λ is a partition for a representation and η is a partition for a conjugacy class

$$f_\lambda(\eta) = \binom{|\lambda|}{|\eta|}|C_\eta|\frac{\chi_\lambda(\eta + [1]^{|\lambda|-|\eta|})}{d_\lambda} \tag{19}$$

$|C_\eta|$ is the size of the conjugacy class represented by the partition η.

Then, including ζ part

$$\mathbf{p}_k(\lambda) = k f_\lambda(\overline{(k)}) \tag{20}$$

Remember that

$$|C_{[k,1^{n-k}]}| = \frac{n!}{k(n-k)!} = \binom{n}{k}(k-1)! \tag{21}$$

We can also write the completed cycles as sums of cut-and-join operators so that

$$f_\lambda(\overline{(k)}) = \frac{\chi_\lambda(\Sigma_{\overline{[k]}})}{d_\lambda} \tag{22}$$

where

$$\Sigma_{\overline{[k]}} = \Sigma_{[k]} + \cdots \tag{23}$$

The first few terms are

$$\Sigma_{\overline{[1]}} = n - \frac{1}{24} \tag{24}$$

$$\Sigma_{\overline{[2]}} = \Sigma_{[2]} \tag{25}$$

$$\Sigma_{\overline{[3]}} = \Sigma_{[3]} + \frac{n(6n-5)}{12} + \frac{7}{2880} \tag{26}$$

$$\Sigma_{\overline{[4]}} = \Sigma_{[4]} + \frac{8n-11}{4}\Sigma_{[2]} \tag{27}$$

C From $\zeta(-k)$ to $\mathrm{vol}(U(N))$

Provides vacuum partition function.

Note that in terms of Bernoulli numbers $\zeta(-k) = -\frac{B_{k+1}}{k+1}$ [related to partition function of $c=1$?]

For $g \geq 2$

$$\chi(\mathcal{M}_g) = \frac{B_{2g}}{2g(2g-2)} = -\frac{\zeta(1-2g)}{2g-2} \tag{28}$$

Expressions for $g = 0, 1$ are special.

These factors of $\chi(\mathcal{M}_g)$ also appear in the $c=1$ vacuum partition function, cf. Witten "The N Matrix Model And Gauged WZW Models" [3] and for resolved conifold in Ooguri-Vafa [8].

The relation to $\mathrm{vol}(U(N))$ is

$$\log(\mathrm{vol}(U(N))) = -\sum_g \frac{\chi(\mathcal{M}_g)}{N^{2g-2}} \tag{29}$$

D Failed attempt to interpret OP_k versus C_k as renormalisation of $1/N$

If we sum the $OP_k(R)$ with a coefficient $\frac{1}{M}$ to keep track and resum to get coefficients of $C_p(R)$

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{(-1)^k}{kM^k} OP_k(R) &= \sum_{p=1}^{\infty} \frac{(-1)^p}{M^p} C_p(R) \sum_{i=0}^{\infty} \frac{1}{(2M)^{2i}} \frac{1}{p+2i} \binom{p+2i}{2i+1} \\
&= \sum_{p=1}^{\infty} \frac{(-1)^p}{M^p} C_p(R) \sum_{i=0}^{\infty} \frac{1}{(2M)^{2i}} \frac{1}{p-1} \binom{p+2i-1}{2i+1} \\
&= \sum_{p=1}^{\infty} \frac{(-1)^p}{(p-1)M^p} C_p(R) M \left[\frac{1}{(1-\frac{1}{2M})^{p-1}} - \frac{1}{(1+\frac{1}{2M})^{p-1}} \right] \\
&= \sum_{p=1}^{\infty} \frac{(-1)^p}{(p-1)M^{p-1}} C_p(R) \left[\frac{1}{(1-\frac{1}{2M})^{p-1}} - \frac{1}{(1+\frac{1}{2M})^{p-1}} \right]
\end{aligned} \tag{30}$$

This looks roughly like a power-by-power renormalisation of the coupling

$$\frac{1}{N^p} = \frac{1}{M^p} \left[\frac{1}{(1-\frac{1}{2M})^p} - \frac{1}{(1+\frac{1}{2M})^p} \right] \tag{31}$$

to agree with the $\dim_N R$ expression.

Problems: includes $C_1(R)$, LHS is not analogue of exponent of $\dim_N R$ exactly.

To get correct leading terms of C_k in expansion of OP_k try

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{(-1)^k}{k(k-1)M^{k-1}}OP_k(R) &= \sum_{p=1}^{\infty}\frac{(-1)^p}{M^{p-1}}C_p(R)\sum_{i=0}^{\infty}\frac{1}{(2M)^{2i}}\frac{1}{(p+2i)(p+2i-1)}\binom{p+2i}{2i+1}\\ &= \sum_{p=1}^{\infty}\frac{(-1)^p}{M^{p}}C_p(R)\sum_{i=0}^{\infty}\frac{1}{(2M)^{2i}}\frac{1}{(p-1)(p-2)}\binom{p+2i-2}{2i+1}\\ &= \sum_{p=1}^{\infty}\frac{(-1)^p}{(p-1)M^{p}}C_p(R)\frac{M}{p-2}\left[\frac{1}{(1-\frac{1}{2M})^{p-2}}-\frac{1}{(1+\frac{1}{2M})^{p-2}}\right]\end{aligned} \tag{32}$$

NB: second line illegal for $p \leq 2$.

Also tried pulling out renormalisation for C_2 and then multiplying - didn't work.

E Other relevant places in literature

The $\mathbf{p}_k(\lambda)$ appear all over the topological strings literature.

Eynard "All orders asymptotic expansion of large partitions" [10].

Alexandrov "Matrix Models for Random Partitions" [11].

Nekrasov $U(1)$ gauge theory to $\mathbb{CP}^1$ conjecture. Toda lattice also appears. Get free fermions.

Brini recent.

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