Energy Systems, Summer Semester 2025
Lecture 9: Complex Markets

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Optimisation Revision

Optimisation problem

We have an **objective function** $f : \mathbb{R}^k \to \mathbb{R}$

$$\max_x f(x)$$

$[x = (x_1, \dots x_k)]$ subject to some **constraints** within $\mathbb{R}^k$:

$$\begin{array}{cccc} g_i(x) = c_i & \leftrightarrow & \lambda_i & i = 1, \dots n \\ h_j(x) \leq d_j & \leftrightarrow & \mu_j & j = 1, \dots m \end{array}$$

λ_i and μ_j are the **KKT multipliers** we introduce for each constraint equation; they measure the change in the objective value of the optimal solution obtained by relaxing the constraints (for this reason they are also called **shadow prices**).

KKT conditions

The **Karush-Kuhn-Tucker (KKT) conditions** are necessary conditions that an optimal solution x^*, μ^*, λ^* always satisfies (up to some regularity conditions):

1. **Stationarity**: For $l = 1, \ldots k$

$$\frac{\partial \mathcal{L}}{\partial x_l} = \frac{\partial f}{\partial x_l} - \sum_i \lambda_i^* \frac{\partial g_i}{\partial x_l} - \sum_j \mu_j^* \frac{\partial h_j}{\partial x_l} = 0$$

2. **Primal feasibility**:

$$g_i(x^*) = c_i$$
$$h_j(x^*) \leq d_j$$

3. **Dual feasibility**: $\mu_j^* \geq 0$

4. **Complementary slackness**: $\mu_j^*(h_j(x^*) - d_j) = 0$

min/max and signs

If the problem is a **maximisation** problem (e.g. welfare maximisation), then $\mu_j^* \geq 0$ since $\mu_j = \frac{\partial \mathcal{L}}{\partial d_j}$ and if we increase d_j in the constraint $h_j(x) \leq d_j$, then the feasible space can only get bigger. Since if $X \subseteq X'$

$$\max_{x \in X} f(x) \leq \max_{x \in X'} f(x)$$

then the objective value at the optimum point can only get bigger, and thus $\mu_j^* \geq 0$. (If $d_j \to \infty$ then the constraint is no longer binding, if $d_j \to -\infty$ then the feasible space vanishes.)

If however the problem is a **minimisation** problem (e.g. cost minimisation) then we can use

$$\min_{x \in X} f(x) = -\max_{x \in X} [-f(x)]$$

We can keep our definition of the Lagrangian and almost all the KKT conditions, but we have a change of sign $\mu_j^* \leq 0$, since

$$\min_{x \in X} f(x) \geq \min_{x \in X'} f(x)$$

The λ_i^* also change sign.

Welfare maximisation revision

KKT and Welfare Maximisation 1/2

Apply KKT now to maximisation of total economic welfare:

$$\max_{\{d_b\},\{g_s\}} f(\{d_b\},\{g_s\}) = \left[\sum_b U_b(d_b) - \sum_s C_s(g_s)\right]$$

subject to the balance constraint:

$$g(\{d_b\},\{g_s\}) = \sum_b d_b - \sum_s g_s = 0 \qquad \leftrightarrow \qquad \lambda$$

and any other constraints (e.g. limits on generator capacity, etc.).

Our optimisation variables are $\{x\} = \{d_b\} \cup \{g_s\}$.

We get from KKT stationarity at the optimal point:

$$0 = \frac{\partial f}{\partial d_b} - \lambda^* \frac{\partial g}{\partial d_b} = U_b'(d_b^*) - \lambda^* = 0$$

$$0 = \frac{\partial f}{\partial g_s} - \lambda^* \frac{\partial g}{\partial g_s} = -C_s'(g_s^*) + \lambda^* = 0$$

KKT and Welfare Maximisation 2/2

So at the optimal point of maximal total economic welfare we get the same result as if everyone maximises their own welfare separately based on the price λ^*:

$$U_b'(d_b^*) = \lambda^*$$
$$C_s'(g_s^*) = \lambda^*$$

This is the CENTRAL result of microeconomics.

If we have further inequality constraints that are binding (e.g. capacity constraints), then these equations will receive additions with $\mu_i^* > 0$.

Optimise Single Node with Linear Generation Costs and Demand Utility

Simplified world: linear generation costs, linear demand utility

We will now turn to a simpler world: all the generator cost functions are linear

$$C_s(g_s) = o_s g_s$$

and each generator has limited output $0 \leq g_s \leq G_s$. The marginal cost function is a constant $C_s'(g_s) = o_s$.

The quantity G_s and marginal cost o_s define a **supply offer**.

All the consumer utility functions are also linear

$$U_b(d_b) = v_b d_b$$

and each consumer has limited consumption $0 \leq d_b \leq D_b$. The marginal utility function is a constant $U_b'(d_b) = v_b$.

The quantity D_b and marginal utility v_b define a **demand bid**.

Supply-demand linear example: generator offers

Example from Kirschen and Strbac pages 56-58.

The following generators offer into the market for the hour between 0900 and 1000 on 20th April 2016:

Company	Quantity [MW]	Marginal cost [$/MWh]
Red	200	12
Red	50	15
Red	150	20
Green	150	16
Green	50	17
Blue	100	13
Blue	50	18

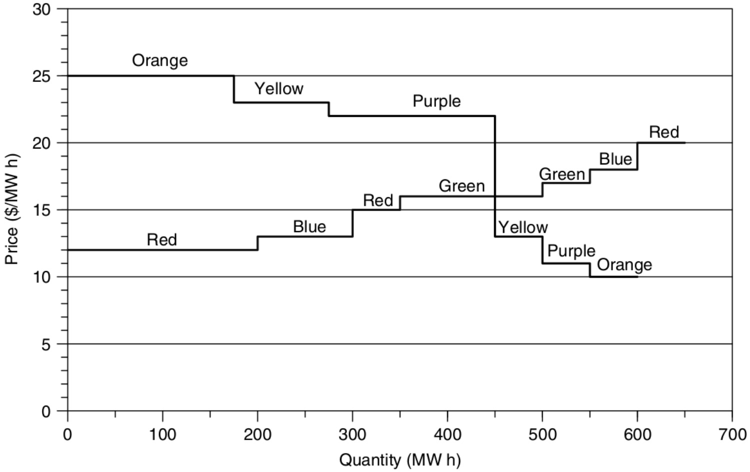
Supply-demand linear example: consumer bids

The following consumers make bids for the same period:

Company	Quantity [MW]	Marginal utility [$/MWh]
Yellow	50	13
Yellow	100	23
Purple	50	11
Purple	150	22
Orange	50	10
Orange	200	25

Supply-demand example: Curve

If the bids and offers are stacked up in order, the supply and demand curves meet with a demand of 450 MW at a system marginal price of $\lambda^* = 16$ \$/MWh.



Source: Kirschen & Strbac

Supply-demand example: Revenue and Expenses

Dispatch and revenue/expense of each company:

Company	Production [MWh]	Consumption [MWh]	Revenue [$]	Expense [$]
Red	250		4000	
Blue	100		1600	
Green	100		1600	
Orange		200		3200
Yellow		100		1600
Purple		150		2400
Total	450	450	7200	7200

For the analysis of the KKT equations, we will simplify even further.

We consider a single demand bid of volume D so that the demand does not respond to price changes (i.e. the demand is **perfectly inelastic**) up to a very high marginal utility $v >> o_s \; \forall s$, i.e.

$$U(d) = vd$$

for $d \leq D$.

v is sometimes called the **Value Of Lost Load (VOLL)**.

In this case we get for our welfare maximisation:

$$\max_{d,\{g_s\}} \left[vd - \sum_s o_s g_s \right]$$

subject to:

$$\begin{aligned} d - \sum_s g_s &= 0 & \leftrightarrow \quad & \lambda \\ d &\leq D & \leftrightarrow \quad & \mu \\ g_s &\leq G_s & \leftrightarrow \quad & \bar{\mu}_s \\ -g_s &\leq 0 & \leftrightarrow \quad & \underline{\mu}_s \end{aligned}$$

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Suppose all generators have the same marginal cost o and we represent their total dispatch by g and total capacity by G

$$\max_{d,g} [vd - og]$$

such that:

$$
\begin{aligned}
d - g = 0 &\qquad \leftrightarrow \qquad \lambda \\
d \leq D &\qquad \leftrightarrow \qquad \mu \\
g \leq G &\qquad \leftrightarrow \qquad \bar{\mu} \\
-g \leq 0 &\qquad \leftrightarrow \qquad \underline{\mu}
\end{aligned}
$$

Simplest example: one generator type, perfectly inelastic demand

If $D < G$ then since $v >> o$, it will be always be welfare-maximising to dispatch to satisfy the load, i.e.

$$g^* = d^* = D$$

If the demand is non-zero then since $g^* > 0$ by complementarity we have $\underline{\mu}^* = 0$. Since $D < G$ then $g^* < G$ and by complementarity we have $\bar{\mu}^* = 0$. To compute λ^* we use stationarity:

$$0 = \frac{\partial \mathcal{L}}{\partial g} = \frac{\partial f}{\partial g} - \sum_i \lambda_i^* \frac{\partial g_i}{\partial g} - \sum_j \mu_j^* \frac{\partial h_j}{\partial g} = -o + \lambda^* - \bar{\mu}^* + \underline{\mu}^*$$

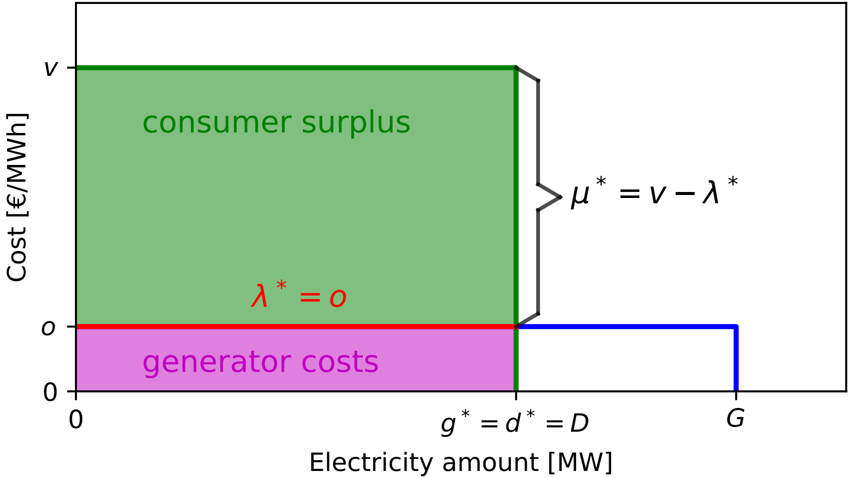
Thus $\lambda^* = o$, which is the cost per unit of supplying extra demand. The **generator sets the price**. There is no generator short-term profit and a large consumer surplus.

For the load μ^* can be non-zero because $d^* = D$:

$$0 = \frac{\partial \mathcal{L}}{\partial d} = \frac{\partial f}{\partial d} - \sum_i \lambda_i^* \frac{\partial g_i}{\partial d} - \sum_j \mu_j^* \frac{\partial h_j}{\partial d} = v - \lambda^* - \mu^*$$

$\mu^* = v - \lambda^*$ is the marginal benefit of each increase in demand.

For the case $D < G$:



If $D > G$ then the generator will dispatch up to its maximum capacity

$$g^* = d^* = G$$

For its lower limit we have $\underline{\mu}^* = 0$. From stationarity:

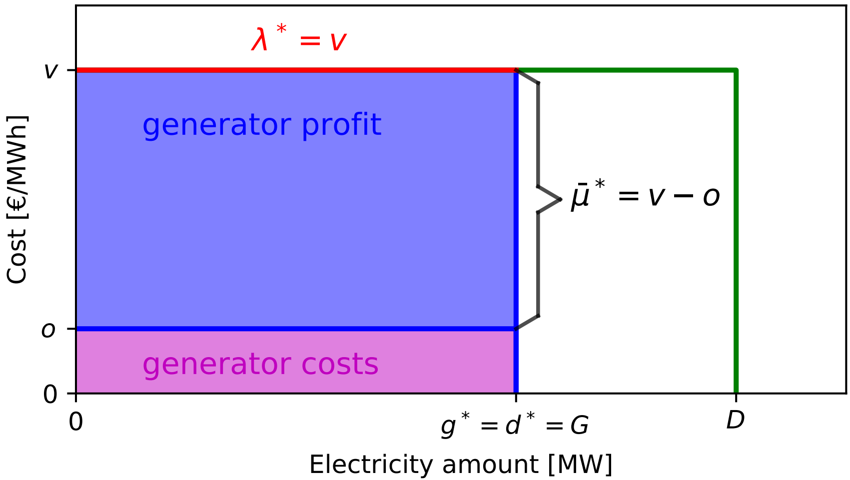
$$0 = \frac{\partial \mathcal{L}}{\partial g} = \frac{\partial f}{\partial g} - \sum_i \lambda_i^* \frac{\partial g_i}{\partial g} - \sum_j \mu_j^* \frac{\partial h_j}{\partial g} = -o + \lambda^* - \bar{\mu}^* + \underline{\mu}^*$$

Thus $\lambda^* = o + \bar{\mu}^*$. To find λ^* we have to look at the demand:

$$0 = \frac{\partial \mathcal{L}}{\partial d} = \frac{\partial f}{\partial d} - \sum_i \lambda_i^* \frac{\partial g_i}{\partial d} - \sum_j \mu_j^* \frac{\partial h_j}{\partial d} = v - \lambda^* - \mu^*$$

Since $d^* < D$, $\mu^* = 0$, $\lambda^* = v$ and thus $\bar{\mu}^* = v - o$, which is the marginal benefit of increasing the generator capacity G. The **demand sets the price**. There is no consumer surplus and the generator makes a large profit. $\bar{\mu}^*$ is the **inframarginal rent**, i.e. the difference between the market price and the generator's marginal cost. It is also know as the **contribution margin** towards paying the fixed costs of the generator.

For the case $D > G$:



Next simplest example: several generators, fixed demand

Suppose we have several generators with dispatch g_s and strictly ordered operating costs o_s such that $o_s < o_{s+1}$. We now maximise

$$\max_{\{d,g_s\}} \left[vd - \sum_s o_s g_s \right]$$

such that

$$\begin{aligned} d - \sum_s g_s = 0 \quad &\leftrightarrow \quad \lambda \\ d \leq D \quad &\leftrightarrow \quad \mu \\ g_s \leq G_s \quad &\leftrightarrow \quad \bar{\mu}_s \\ -g_s \leq 0 \quad &\leftrightarrow \quad \underline{\mu}_s \end{aligned}$$

Stationarity gives us for each generator g_s:

$$0 = \frac{\partial \mathcal{L}}{\partial g_s} = -o_s + \lambda^* - \bar{\mu}_s^* + \underline{\mu}_s^*$$

and from complementarity we get

$$\bar{\mu}_s(g_s^* - G_s) = 0 \qquad \underline{\mu}_s g_s^* = 0$$

We can see by inspection that we will dispatch the cheapest generation first. Suppose that we have enough generation for the demand, i.e. $D < \sum_s G_s$. [If $D > \sum_s G_s$ we have the same situation as for a single generator, i.e. $\lambda^* = v$, so that the demand sets the price.]

Find the generator m on the margin where the supply curve intersects with the demand D, i.e. the m where $\sum_{s=1}^{m-1} G_s < D < \sum_{s=1}^{m} G_s$.

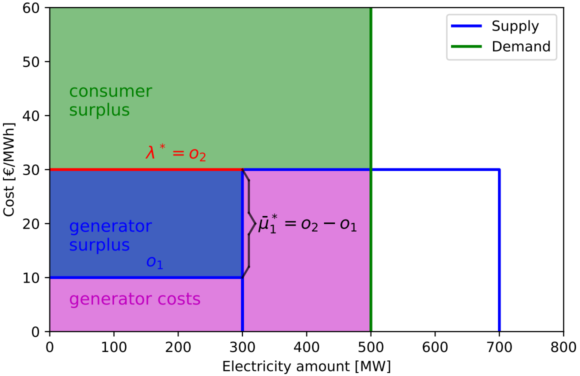
For $s \leq m-1$ we have $g_s^* = G_s$, $\underline{\mu}_s^* = 0$, $\bar{\mu}_s^* = \lambda^* - o_s$. $\bar{\mu}_s^*$ are the **inframarginal rents**.

For $s = m$ we have $g_m^* = D - \sum_{s=1}^{m-1} G_s$ to cover what's left of the demand. Since $0 < g_m^* < G_m$ we have $\underline{\mu}_m^* = \bar{\mu}_m^* = 0$ and thus $\lambda^* = o_m$.

Next simplest example: several generators, fixed demand

Specific example of two generators with $G_1 = 300$ MW, $G_2 = 400$ MW, $o_1 = 10$ €/MWh, $o_2 = 30$ €/MWh and $D = 500$ MW.

In this case $m = 2$, $g_1^* = G_1 = 300$ MW, $g_2^* = d - G_1 = 200$ MW, $\lambda^* = o_2$, $\underline{\mu}_i = 0$, $\bar{\mu}_2 = 0$ and $\bar{\mu}_1 = o_2 - o_1$.



From welfare maximisation to cost minimisation

For the case $D > \sum_s G_s$ we can instead imagine that the demand is rigidly fixed to D and that instead we have a dummy generator with dispatch $g_d = D - \sum_s G_s$ that represents **load shedding**. In this case we can substitute $d = D - g_d$ to get

$$\max_{\{g_d, g_s\}} \left[vD - vg_d - \sum_s o_s g_s \right]$$

such that

$$D - g_d - \sum_s g_s = 0 \qquad \leftrightarrow \qquad \lambda$$

$$g_s \leq G_s \qquad \leftrightarrow \qquad \bar{\mu}_s$$

$$-g_s \leq 0 \qquad \leftrightarrow \qquad \underline{\mu}_s$$

Since vD is a constant, we can use $\max_{x \in X} [-f(x)] = -\min_{x \in X} f(x)$ to recast this as a minimisation of the total generator costs, absorbing g_d into the set $\{g_s\}$. The constant vD is dropped.

From welfare maximisation to cost minimisation

We have turned the maximisation of total welfare into **cost minimisation**:

$$\min_{\{g_s\}} \sum_s o_s g_s$$

such that:

$$\sum_s g_s - d = 0 \qquad \leftrightarrow \qquad \lambda$$

$$g_s \leq G_s \qquad \leftrightarrow \qquad \bar{\mu}_s$$

$$-g_s \leq 0 \qquad \leftrightarrow \qquad \underline{\mu}_s$$

The most expensive generator has $o_s = v$ and $G_s = \infty$ and represents **load shedding**.

We've replaced the symbol D with d for simplicity going forward (d is now a constant).

NB: Because the signs of the KKT multipliers change when we go from maximisation to minimisation, we've also changed the sign of the balance constraint to keep the marginal price λ positive.

Optimise nodes in a network

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Now let's suppose we have several nodes i with different loads and different generators, with flows f_ℓ in the network lines ℓ.

Now we have additional optimisation variables f_ℓ AND additional constraints for welfare maximisation:

$$\max_{\{d_{i,b}\},\{g_{i,s}\},\{f_\ell\}} \left[\sum_{i,b} U_{i,b}(d_{i,b}) - \sum_{i,s} C_{i,s}(g_{i,s}) \right]$$

such that demand is met either by generation or by the network at each node i

$$\sum_b d_{i,b} - \sum_s g_{i,s} + \sum_\ell K_{i\ell} f_\ell = 0 \qquad \leftrightarrow \qquad \lambda_i$$

Note there is now a **market price for each node**. As before, generator constraints are satisified

$$g_{i,s} \leq G_{i,s} \qquad \leftrightarrow \qquad \bar{\mu}_{i,s}$$
$$-g_{i,s} \leq 0 \qquad \leftrightarrow \qquad \underline{\mu}_{i,s}$$

Linear cost minimisation at several nodes in a network

For cost minimisation we have a fixed load d_i at each node, and absorb load-shedding above a value v into a dummy generator.

Now we minimise over f_ℓ and $g_{i,s}$ for the case of linear cost functions:

$$\min_{\{g_{i,s}\},\{f_\ell\}} \sum_{i,s} o_{i,s} g_{i,s}$$

such that demand is met either by generation or by the network at each node i

$$\sum_s g_{i,s} - d_i = \sum_\ell K_{i\ell} f_\ell \qquad \leftrightarrow \qquad \lambda_i$$

and generator constraints are satisified

$$g_{i,s} \leq G_{i,s} \qquad \leftrightarrow \qquad \bar{\mu}_{i,s}$$
$$-g_{i,s} \leq 0 \qquad \leftrightarrow \qquad \underline{\mu}_{i,s}$$

In addition we have constraints on the line flows.

First, they have to satisfy Kirchoff's Voltage Law around each closed cycle c:

$$\sum_c C_{\ell c} x_\ell f_\ell = 0 \qquad \leftrightarrow \qquad \lambda_c$$

and in addition the flows cannot overload the thermal limits, $|f_\ell| \leq F_\ell$

$$f_\ell \leq F_\ell \qquad \leftrightarrow \qquad \bar{\mu}_\ell$$
$$-f_\ell \leq F_\ell \qquad \leftrightarrow \qquad \underline{\mu}_\ell$$

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At node 1 we have demand of $d_1 = 100$ MW and a generator with costs $o_1 = 10$ €/MWh and a capacity of $G_1 = 300$ MW.

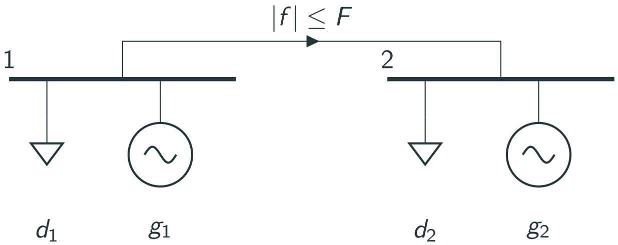
At node 2 we have demand of $d_2 = 100$ MW and a generator with costs $o_2 = 20$ €/MWh and a capacity of $G_2 = 300$ MW.

What happens if the capacity of the line connecting them is $F = 0$?

What about $F = 50$ MW?

What about $F = \infty$?

Simplest example: two nodes connected by a single line



$$g_1 - d_1 = f \quad \leftrightarrow \quad \lambda_1 \qquad\qquad g_2 - d_2 = -f \quad \leftrightarrow \quad \lambda_2$$

$$d_1 = 100 \text{ MW} \qquad d_2 = 100 \text{ MW}$$
$$G_1 = 300 \text{ MW} \qquad G_2 = 300 \text{ MW}$$
$$o_1 = 10 \text{ €/MWh} \qquad o_2 = 20 \text{ €/MWh}$$

Simplest example: two nodes connected by a single line

Out optimisation problem has objective function:

$$\min_{g_1, g_2, f} [o_1 g_1 + o_2 g_2]$$

subject to the following constraints:

$$
\begin{aligned}
g_1 - d_1 &= f & \leftrightarrow & \quad \lambda_1 \\
g_2 - d_2 &= -f & \leftrightarrow & \quad \lambda_2 \\
g_1 &\leq G_1 & \leftrightarrow & \quad \bar{\mu}_1 \\
-g_1 &\leq 0 & \leftrightarrow & \quad \underline{\mu}_1 \\
g_2 &\leq G_2 & \leftrightarrow & \quad \bar{\mu}_2 \\
-g_2 &\leq 0 & \leftrightarrow & \quad \underline{\mu}_2 \\
f &\leq F & \leftrightarrow & \quad \bar{\mu} \\
-f &\leq F & \leftrightarrow & \quad \underline{\mu}
\end{aligned}
$$

Two nodes: Case $F = 0$

For the case $F = 0$ the nodes are like two separated islands, $f^* = 0$.

The generator on each island provides the demand separately, so:

$$g_1^* = d_1 \qquad \text{and} \qquad g_2^* = d_2$$

Neither generator has any binding constraints, since in each case the demand (100 MW) is less than the generator capacity (300 MW), so

$$\bar{\mu}_1^* = \underline{\mu}_1^* = \bar{\mu}_2^* = \underline{\mu}_2^* = 0$$

From stationarity for each site we get

$$0 = \frac{\partial \mathcal{L}}{\partial g_i} = o_i - \lambda_i^* - \bar{\mu}_i^* + \underline{\mu}_i^*$$

Thus we have at each site $\lambda_i^* = o_i$, as if we had optimised the nodes separately.

Two nodes: Case $F = 50$ MW

For the case F =50 MW the cheaper node 1 will export to the more expensive node 2 as much as the restricted capacity F allows:

$$f^* = F = 50 \text{ MW}$$

Generator 1 covers 50 MW of the demand from node 2:

$$g_1^* = d_1 + f^* = 150 \text{ MW} \qquad \text{and} \qquad g_2^* = d_2 - f^* = 50 \text{ MW}$$

Neither generator has any binding constraints, so

$$\bar{\mu}_1^* = \underline{\mu}_1^* = \bar{\mu}_2^* = \underline{\mu}_2^* = 0$$

and thus we have again different prices at each $\lambda_i^* = o_i$. For the flow:

$$0 = \frac{\partial \mathcal{L}}{\partial f} = 0 + \lambda_1^* - \lambda_2^* - \bar{\mu}^* + \underline{\mu}^*$$

Only the upper limit is binding, so we get $\underline{\mu}^* = 0$ and $\bar{\mu}^* = \lambda_1^* - \lambda_2^* = o_1 - o_2 = -10$ €/MWh.

$\bar{\mu}^*$ is the cost reduction if we expand the transmission capacity F by ε, allowing us to substitute some of the expensive generation at node 2 with cheap generation from node 1.

For the case $F = \infty$ we have unrestricted capacity, so it is like merging the two nodes to one node. Now all the demand is covered by the cheapest node:

$$f^* = d_2 = 100 \text{ MW}$$

Generator 1 covers all the demand:

$$g_1^* = d_1 + d_2 = 200 \text{ MW} \qquad \text{and} \qquad g_2^* = 0$$

Only generator 2 has a non-zero KKT multiplier, so at node 1 we have $\lambda_1^* = o_1$ and at node 2 we have:

$$\underline{\mu}_2^* = \lambda_2^* - o_2$$

From KKT for the flow f we have no constraints so $\bar{\mu}^* = \underline{\mu}^* = 0$ and from stationarity

$$0 = \frac{\partial \mathcal{L}}{\partial f} = 0 + \lambda_1^* - \lambda_2^* - \bar{\mu}^* + \underline{\mu}^*$$

i.e. $\lambda_1^* = \lambda_2^*$. We have price equalisation, as if it were a single node.

Two node: demand payments versus generation revenue

Now let's compare for our examples what each demand pays $\lambda_i^* d_i$ and what each generator receives as revenue $\lambda_i^* g_i^*$ from each market.

Case	λ_1^* [€/MWh]	λ_2^* [€/MWh]	$\lambda_1^* d_1$ [€/h]	$\lambda_2^* d_2$ [€/h]	$\sum_i \lambda_i^* d_i$ [€/h]	$\lambda_1^* g_1^*$ [€/h]	$\lambda_2^* g_2^*$ [€/h]	$\sum_i \lambda_i^* g_i^*$ [€/h]
$F = 0$	10	20	1000	2000	3000	1000	2000	3000
$F = 50$	10	20	1000	2000	3000	1500	1000	2500
$F = \infty$	10	10	1000	1000	2000	2000	0	2000

NB: In the case with $F = 50$, total demand payments are 3000 €/h, whereas the generators are only receiving 2500 €/h.

Where is the missing money (500 €/h) going?

Answer: to the network operator for service of doing arbitrage, buying low and selling high.

Congestion rent

Due to the congestion of the transmission line, the marginal cost of producing electricity can be different at node 1 and node 2. The competitive price at node 2 is higher than at node 1 – this corresponds to **locational marginal pricing**, or **nodal pricing**.

Since consumers pay and generators get paid the price in their local market, in case of congestion there is a difference between the total payment of consumers and the total revenue of producers – this is the **merchandising surplus** or **congestion rent**, collected by the network operator. For each line it is given by the price difference in both regions times the amount of power flow between them:

$$\text{Congestion rent} = \Delta\lambda \times f$$

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Returning to our two node example:

Case	Demand pays [€/h]	Generator gets [€/h]	$\lambda_2^* - \lambda_1^*$ [€/MWh]	flow f [MW]	Cong. rent [€/h]
$F = 0$	3000	3000	10	0	0
$F = 50$	3000	2500	10	50	500
$F = \infty$	2000	2000	0	100	0

To get a congestion rent, we need congestion somewhere in the network to cause a price difference between the nodes, as well as a non-zero flow between the nodes.

In this example we saw that the sum of what consumers pay does not always equal the sum of generator revenue.

In fact if we take the balance constraint and sum it weighted by the market price at each node we find

$$\sum_i \lambda_i^* d_i - \sum_i \lambda_i^* \sum_s g_{i,s}^* = -\sum_i \lambda_i^* \sum_\ell K_{i\ell} f_\ell^*$$

The quantity for each ℓ

$$-f_\ell^* \sum_i K_{i\ell} \lambda_i^* = f_\ell(\lambda_{\mathrm{end}}^* - \lambda_{\mathrm{start}}^*)$$

is called the **congestion rent** and is the money the network operator receives for transferring power from a low price node (start) to a high price node (end), 'buy it low, sell it high'.

It is zero if: a) the flow is zero or b) the price difference is zero.

Two nodes, quadratic cost function

$$\min_{g_1, g_2, f} \left[C_1(g_1) + C_2(g_2) \right]$$

$$f \leq F \leftrightarrow \bar{\mu} \qquad -f \leq F \leftrightarrow \underline{\mu}$$

$$d_1 = 500 \qquad g_1 \qquad d_2 = 1500 \qquad g_2$$

$$C_1'(g_1) = 10 + 0.01 g_1 \qquad C_2'(g_2) = 13 + 0.02 g_2$$

$$g_1 - d_1 = f \leftrightarrow \lambda_1 \qquad g_2 - d_2 = -f \leftrightarrow \lambda_2$$

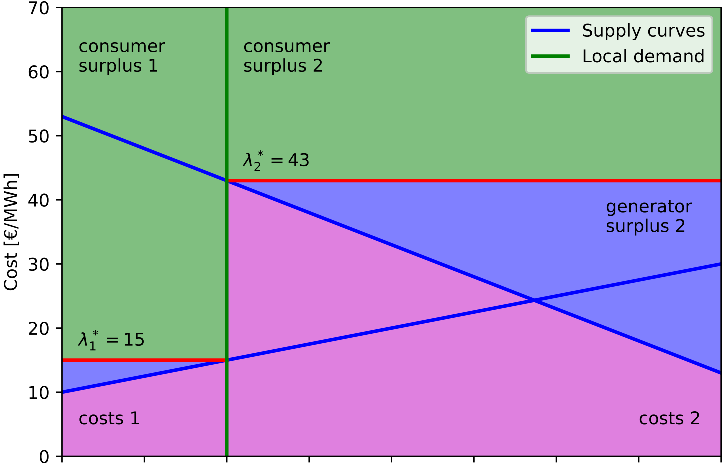
Two nodes, quadratic cost function

From stationarity

$$
\begin{aligned}
0 &= \frac{\partial \mathcal{L}}{\partial g_1} = C_1'(g_1^*) - \lambda_1^* \\
0 &= \frac{\partial \mathcal{L}}{\partial g_2} = C_2'(g_2^*) - \lambda_2^* \\
0 &= \frac{\partial \mathcal{L}}{\partial f} = 0 + \lambda_1^* - \lambda_2^* - \bar{\mu}^* + \underline{\mu}^*
\end{aligned}
$$

Outcome with no transmission capacity $F = 0$

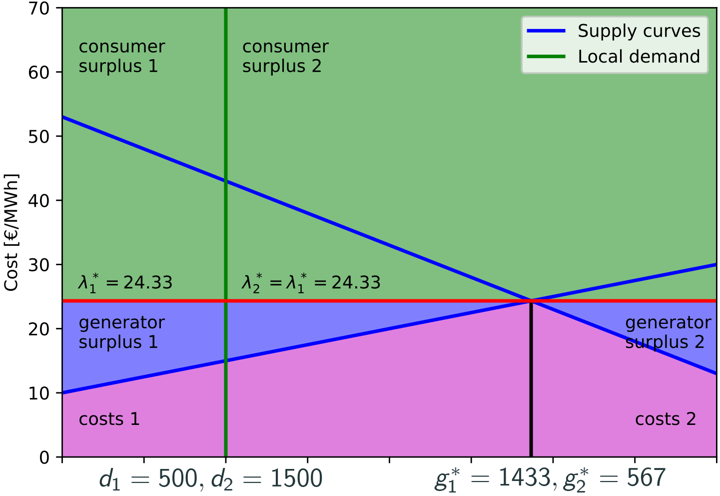
$$F = 0, f^* = 0, g_1^* = 500, g_2^* = 1500, \lambda_1^* = 15, \lambda_2^* = 43, \underline{\mu}^* = 0, \bar{\mu}^* = \lambda_1^* - \lambda_2^* = -28$$



$d_1 = 500, \; d_2 = 1500$
$g_1^* = 500, \; g_2^* = 1500$

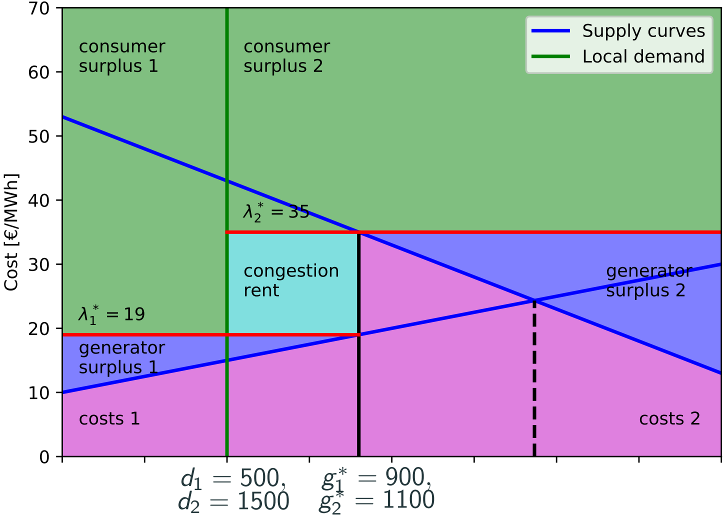
Outcome with unlimited transmission capacity $F = \infty$

$$F = \infty, f^* = 933, g_1^* = 1433, g_2^* = 567, \lambda_1^* = \lambda_2^* = 24.33, \underline{\mu}^* = \bar{\mu}^* = 0$$



Outcome with constrained transmission capacity $F = 400$

$$F = 400, f^* = 400, g_1^* = 900, g_2^* = 1100, \lambda_1^* = 19, \lambda_2^* = 35, \underline{\mu}^* = 0, \bar{\mu}^* = \lambda_1^* - \lambda_2^* = -16$$



Comparison

	Separate markets	Single market	Constrained market
d_1 [MW]	500	500	500
g_1^* [MW]	500	1433	900
λ_1^* [€/MWh]	15	24.33	19
d_2 [MW]	1500	1500	1500
g_2^* [MW]	1500	567	1100
λ_2^* [€/MWh]	43	24.33	35
f^* [MW]	0	933	400
$\underline{\mu}^*$ [€/MWh]	0	0	0
$\bar{\mu}^*$ [€/MWh]	-28	0	-16
$\sum_s \lambda_s \times g_s$ [€]	72000	48660	55600
$\sum_s \lambda_s \times d_s$ [€]	72000	48660	62000
congestion rent	0	0	6400

Advanced: 3-node example

PTDF formulation of linearised optimal power flow

For the 3-node example it is easier to switch to the **Power Transfer Distribution Factors** (PTDF) formulation of the power flow $f_\ell = \sum_k \text{PTDF}_{\ell k} p_k$. Keep our objective

$$\min_{\{g_{i,s}\},\{f_\ell\}} \sum_{i,s} o_{i,s} g_{i,s}$$

and the same generator constraints and line constraints:

$$f_\ell \leq F_\ell \qquad \leftrightarrow \qquad \bar{\mu}_\ell$$

$$-f_\ell \leq F_\ell \qquad \leftrightarrow \qquad \underline{\mu}_\ell$$

but instead of energy conservation at each node and cycle constraints we have:

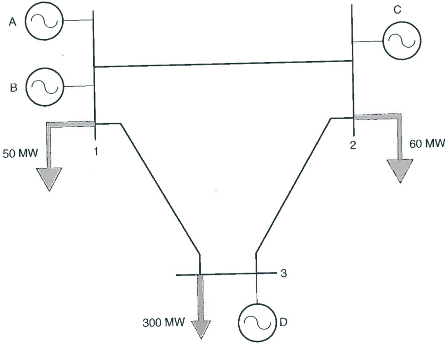
$$f_\ell = \sum_k \text{PTDF}_{\ell k} p_k = \sum_k \text{PTDF}_{\ell k} \left(\sum_s g_{k,s} - d_k \right)$$

and one overall conservation constraint:

$$\sum_i p_i = \sum_i \left(\sum_s g_{i,s} - d_i \right) = 0$$

An example 3-node system

The cheapest generators are are at node 1, most demand is at node 3.



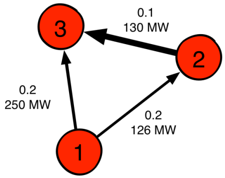
$\sum_i d_i = 410$ MW

Generator	Capacity (MW)	Marginal cost (€/MWh)
A	140	7.5
B	285	6
C	90	14
D	85	10

Line	Reactance (p.u.)	Capacity (MW)
$1 \to 2$	0.2	126
$1 \to 3$	0.2	250
$2 \to 3$	0.1	130

Source: Kirschen and Strbac

Power flows and feasible injections



$$\textbf{PTDF} = \begin{matrix} 1 \to 2 \\ 1 \to 3 \\ 2 \to 3 \end{matrix} \begin{pmatrix} 2/5 & -1/5 & 0 \\ 3/5 & 1/5 & 0 \\ 2/5 & 4/5 & 0 \end{pmatrix}$$

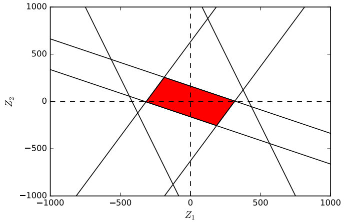
Power flows:

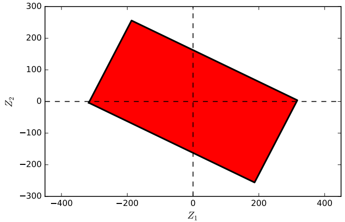
$$f_{1\to 2} = \frac{1}{5}(2p_1 - p_2)$$

$$f_{1\to 3} = \frac{1}{5}(3p_1 + p_2)$$

$$f_{2\to 3} = \frac{2}{5}(p_1 + 2p_2)$$

Implications of six constraints $|f_\ell| \leq F_\ell$ on p_1 and p_2 (p_3 is not independent since $\sum_i p_i = 0$; graphic Z_i is p_i): feasible space is rectangle.



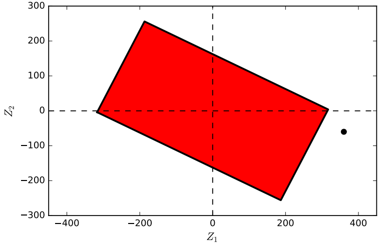


Economic dispatch (ignore grid)

If we ignore the grid constraints, we dispatch the cheapest generators first, market price set by marginal generator A: $\lambda = o_A = 7.5$ €/MWh.

$$g_A = 125 \text{ MW}$$
$$g_B = 285 \text{ MW}$$
$$g_C = g_D = 0$$

$$F_{1\to 2} = 126 \text{ MW}$$
$$F_{1\to 3} = 250 \text{ MW}$$
$$F_{2\to 3} = 130 \text{ MW}$$



$p_1 = 360$ MW, $p_2 = -60$ MW.

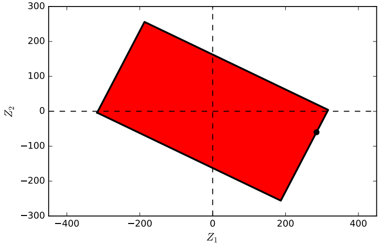
$$f_{1\to 2} = \frac{1}{5}(2p_1 - p_2) = 156 \text{ MW}$$
$$f_{1\to 3} = \frac{1}{5}(3p_1 + p_2) = 204 \text{ MW}$$
$$f_{2\to 3} = \frac{2}{5}(p_1 + 2p_2) = 96 \text{ MW}$$

First line limit is violated! Optimal point (black dot) is outside the red feasible space.

What can we do? Transfer x MW from generator A at node 1 to generator D at node 3 to relieve the line.

$$g_A = 125 \rightarrow g_A = 125 - x$$
$$g_D = 0 \rightarrow g_D = x$$

$$F_{1\rightarrow 2} = 126$$
$$F_{1\rightarrow 3} = 250$$
$$F_{2\rightarrow 3} = 130$$



$$f_{1\rightarrow 2} = \frac{1}{5}(2(p_1 - x) - p_2) = 156 - \frac{2}{5}x = 126$$
$$f_{1\rightarrow 3} = \frac{1}{5}(3p_1 + p_2) = 159$$
$$f_{2\rightarrow 3} = \frac{2}{5}(p_1 + 2p_2) = 66$$

$p_1 = 360 - x, p_2 = -60.$

Solution? $\frac{2}{5}x = 30$, i.e. $x = 75$.

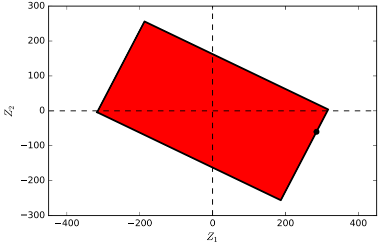
Economic redispatch: solution

Solution with $x = 75$ MW transfer from generator A at node 1 to generator D at node 3 to relieve the line $f_{1\to2}$.

$$g_A = 125 \to g_A = 125 - x = 50$$

$$g_D = 0 \to g_D = x = 75$$

Redispatch cost: 187.5 €/h



$p_1 = 360 - x = 285$, $p_2 = -60$.

$$F_{1\to2} = 126$$

$$F_{1\to3} = 250$$

$$F_{2\to3} = 130$$

$$f_{1\to2} = \frac{1}{5}(2p_1 - p_2) = 126$$

$$f_{1\to3} = \frac{1}{5}(3p_1 + p_2) = 159$$

$$f_{2\to3} = \frac{2}{5}(p_1 + 2p_2) = 66$$

The **nodal marginal price** is equal to the minimal system cost of supplying an additional megawatt of load at this node.

For node 1, there is a cheap generator A to locally supply the demand, so $\lambda_1 = o_A = 7.5$.

For node 3, it cannot be supplied from node 1 because of the transmission constraint on $f_{1\to 2}$, so it must be supplied locally $\lambda_3 = o_D = 10$.

What about node 2? We cannot supply from node 1, node 2's generator C is expensive, so what about from node 3? Suppose we increase demand by ε, i.e. $p_2 \to p_2 - \varepsilon$. Because of the binding transmission constraint $f_{1\to 2} = \frac{1}{5}(2p_1 - p_2) = 126$ we need to compensate by decreasing $p_1 \to p_1 - \frac{1}{2}\varepsilon$. This means we need to increase generator D and $p_3 \to p_3 + \frac{3}{2}\varepsilon$ so we get

$$\lambda_2 = 1.5 o_D - 0.5 o_A = 11.25$$

NB: This price is a composite, not the marginal price of any generator!

Nodal prices

Economic operation of the three-node system using nodal pricing.

	Node 1	Node 2	Node 3	System
Consumption (MW)	50	60	300	410
Production (MW)	335	0	75	410
Nodal marginal price (€/MWh)	7.5	11.25	10	-
Consumer payments (€/h)	375	675	3000	4050
Generator revenue (€/h)	2512.5	0	750	3262.5
Congestion rent (€/h)				787.5

Congestion rent:

Connection	Flow (MW)	'From' price (€/MWh)	'To' price (€/MWh)	Surplus (€/h)
$1 \to 2$	126	7.5	11.25	427.5
$1 \to 3$	159	7.5	10	397.5
$2 \to 3$	66	11.25	10	-82.5
Total				787.5

Note the counter-intuitive flow from node 2 (higher price) to node 3 (lower price)! This is because of grid constraints from KVL.

Note also that the line itself does not have to be congested to have a congestion rent (see e.g. line $1 \to 3$). It suffices that somewhere in the network there is congestion (here: $1 \to 2$).

Example slightly changed

The **nodal marginal price** is equal to the minimal system cost of supplying an additional megawatt of load at this node.

$$g_A = 47.5 \text{ MW}$$
$$g_B = 285 \text{ MW}$$
$$g_C = 0 \text{ MW}$$
$$g_D = 77.5 \text{ MW}$$

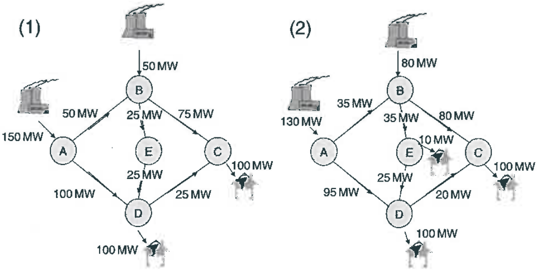
$$F_{1\to 2} = 126 \text{ MW}$$
$$F_{1\to 3} = 250 \text{ MW}$$
$$F_{2\to 3} = \mathbf{65} \text{ MW}$$

Nodal prices:

$$\lambda_1 = o_A = 7.5 \text{ €/MWh}$$
$$\lambda_3 = o_D = 10 \text{ €/MWh}$$
$$\lambda_2 = 2 \times o_A - 1 \times o_D = 5 \text{ €/MWh}$$

$$f_{1\to 2} = \frac{1}{5}(2p_1 - p_2) = 125 \text{ MW}$$
$$f_{1\to 3} = \frac{1}{5}(3p_1 + p_2) = 157.5 \text{ MW}$$
$$f_{2\to 3} = \frac{2}{5}(p_1 + 2p_2) = 65 \text{ MW}$$

The nodal marginal price at node 2 is lower than the marginal cost of any generator!



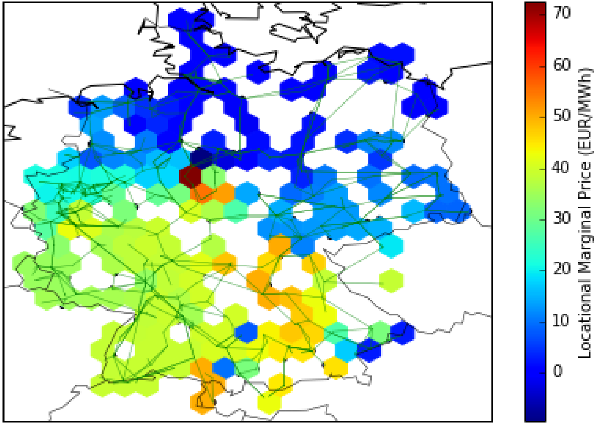
Generator A has marginal costs 60€/MWh, generator B has marginal costs 30€/MWh. The line between E and D is constrained to 25 MW.

The additional load of 10 MW at node E allows cheap generator B to substitute some of expensive generator A and thus **reduces** the system cost by 300 €/h, so $\lambda_E = -30$ €/MWh!

Source: Harris

Nodal prices for Germany

Spot the node with negative price. If we increase the load here, it will relieve a line somewhere that allows more flow from a cheap node to an expensive node, thus reducing the system cost.



Source: PyPSA
(Python for Power System Analysis)

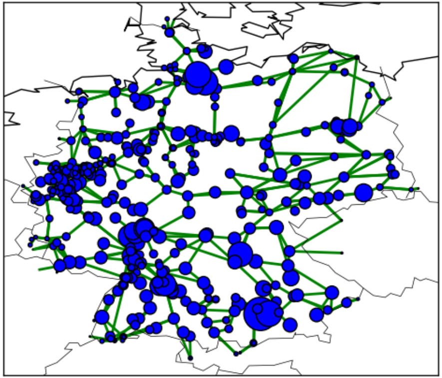
The European Market

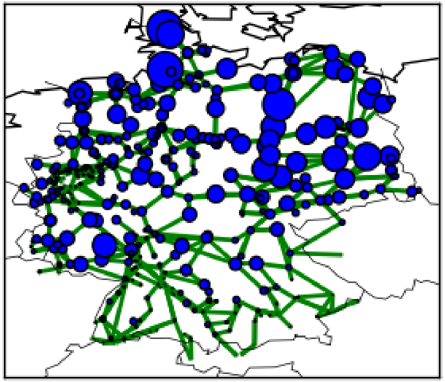
Existing bidding zones

- Bids for German electricity take place in a **giant bidding zone** encompassing both Germany and Luxembourg (Austria was separated from the German bidding zone in October 2018)
- This means that transmission constraints are only visible to the market at the **borders** to the other national zones
- Internal transmission constraints are **ignored** - market bids are handled as if they do not exist
- Only KCL enforced on most borders - KVL much harder

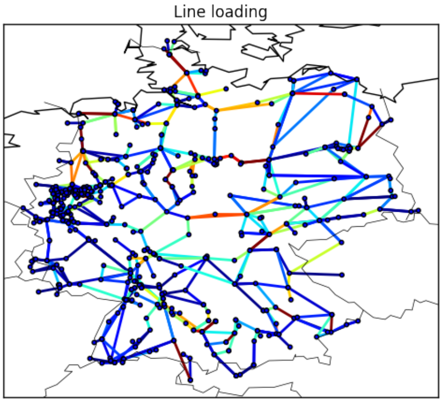
The Problem

Renewables are not always located near demand centres, as in this example from Germany.





The Problem



- This leads to **overloaded lines** in the middle of Germany, which cannot transport all the wind energy from North Germany to the load in South Germany
- It also overloads lines in neighbouring countries due to **loop flows** (unplanned physical flows 'according to least resistance' which do not correspond to traded flows)
- It also **blocks imports and exports** with neighbouring countries, e.g. Denmark

Solution 1: Redispatch after energy market clearing

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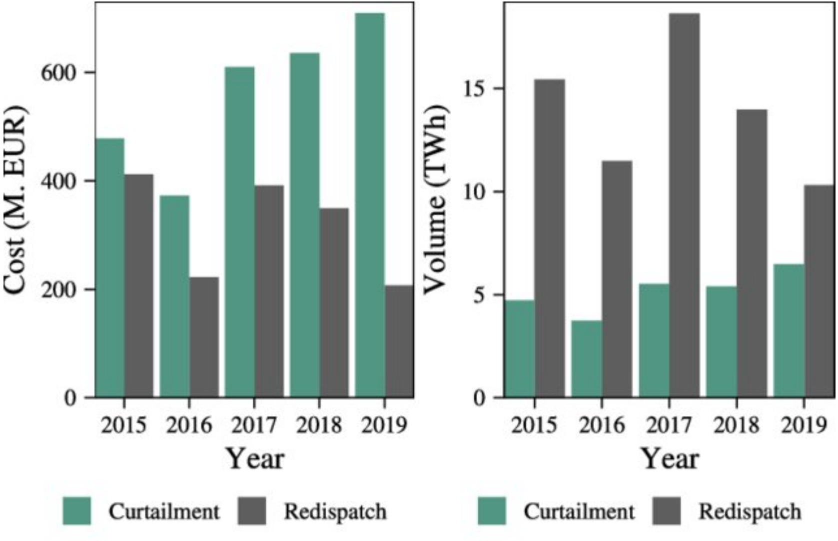
These problems are **not visible** in the day-ahead electricity market, which treats the whole of Germany and Austria as a single bidding zone. It dispatches wind in North Germany as if there was no internal congestion...

To ensure that the physical limits of transmission are not exceeded, the network operator must **'re-dispatch'** power stations and **curtail** (Einspeisemanagement) renewables to restore order. This is **costly** (0.8 redispatch + 0.6 RE-compensation = 1.4 billion EUR in 2017 - although exceptional circumstances in 1st quarter) and results in **lost CO_2-free generation** (5.5 TWh curtailment of RE and CHP in 2017).

International redispatch is sometimes also required (Multilateral Remedial Actions = MRA).

Furthermore, there are **no market incentives** to reinforce the North-South grid, to locate more power stations in South Germany or to build storage / P2X in North Germany.

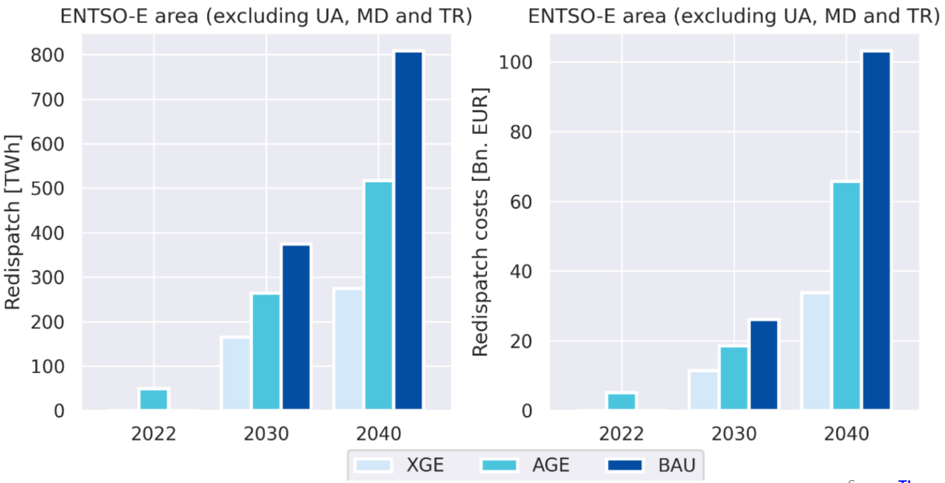
Redispatch in Germany: Trend



(a) Expenditure (b) Volume

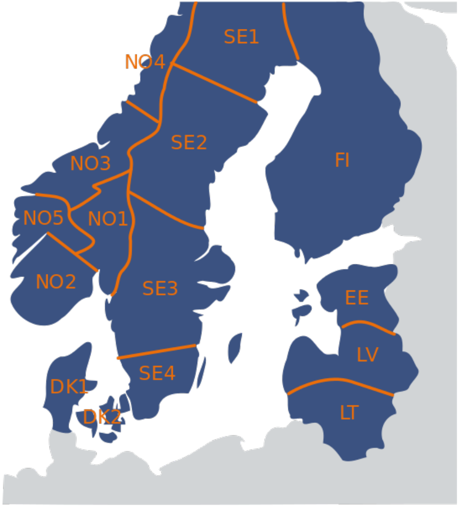
Source: Xiong et al (2021)

The Joint Research Centre (JRC) of the European Commission calculates rising redispatch if the existing bidding zone configuration is kept.



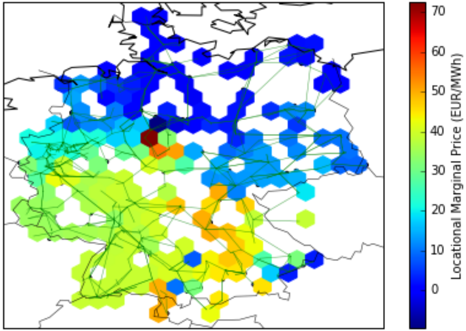
Source: Thomassen et al, 2024

Solution 2: Smaller bidding zones to “see” congested boundaries



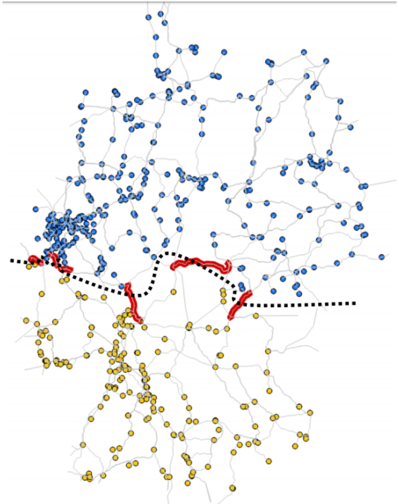
- In Scandinavia they have solved this by introducing **smaller bidding zones**
- Now congestion at the boundaries between zones is taken into account in the **implicit auctions** of the market
- This is also done in Italy (again, a long country), where prices for small consumers are **uniformised** for fairness

Solution 3: Nodal pricing



- The ultimate solution, as used in the US and other markets, is **nodal pricing**, which exposes all transmission congestion
- Considered too complex and subject to market power to be used in Europe, but this is questionable...
- Here we see clearly why many argue for a North-South German split

First step: Split Germany North-South

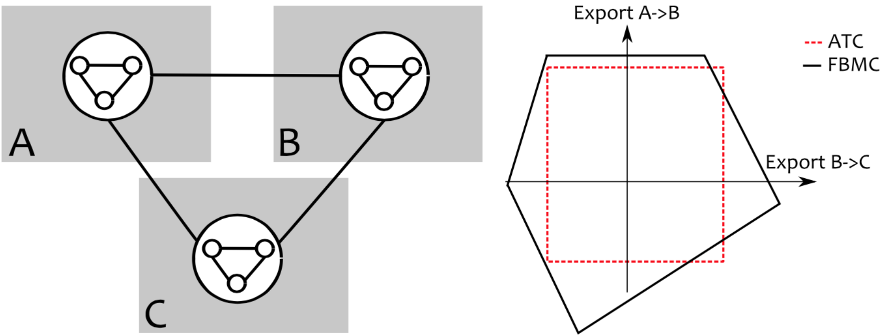


- Initial price difference could average up to 12 EUR/MWh
- Prices would converge with more network expansion
- Redispatch costs reduced by 39% in 2025, 58% in 2035 (assuming NEP 2030 transmission projects get built)
- Politically difficult, may require, like Italy, uniformised price on consumer side

Source: Fraunholz & Hladik, 2018

Solution 1.5: Flow-based market coupling

Flow-based market coupling can be used in zonal markets to see precise individual line constraints via PTDF, instead of “boxing” the feasible space like ATC/NTC schemes do.



(a) Zonal market model.

(b) Flow domain.

Figure 4. In the FMBC method, only one equivalent node per zone is considered, but all (critical) lines are taken into account. In this simple grid, the zonal network consists of 3 nodes and 12 lines. The FBMC flow domain is larger than the ATC flow domain as the physical characteristics of the grid are better represented in the FBMC method.

Source: Van den Bergh, Boury, Delarue

Solution 1.5: Flow-based market coupling

- From May 2015 to June 2022 flow-based market coupling was only applied in Germany, France, Netherlands, Belgium and Luxembourg
- From June 2022 it has extended across northern ‘Core’ countries in continental Europe

Source: JAO Core FBMC

Storage Optimisation

Storage equations

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Now, like the network case where we add different nodes i with different loads, for storage we have to consider different time periods t. This is called **multi-period optimisation**.

Label conventional generators by s, storage by r and now minimise

$$\min_{\{g_{i,s,t}\},\{g_{i,r,t,\text{charge}}\},\{g_{i,r,t,\text{discharge}}\},\{f_{\ell,t}\}} \left[\sum_{i,s,t} o_{i,s} g_{i,s,t} + \sum_{i,r,t} o_{i,r,\text{charge}}\, g_{i,r,t,\text{charge}} + \sum_{i,r,t} o_{i,r,\text{discharge}}\, g_{i,r,t,\text{discharge}}\right]$$

The power balance constraints are now (cf. Lecture 5) for each node i and time t that the demand is met either by generation, storage or network flows:

$$\sum_s g_{i,s,t} + \sum_r (g_{i,r,t,\text{discharge}} - g_{i,r,t,\text{charge}}) - d_{i,t} = \sum_\ell K_{i\ell} f_{\ell,t} \quad \leftrightarrow \quad \lambda_{i,t}$$

Now we have a market price $\lambda_{i,t}$ for each node i and time t.

Storage equations

We have constraints on normal generators

$$0 \leq g_{i,s,t} \leq G_{i,s} \qquad \leftrightarrow \qquad \mu_{i,s,t}$$

and on the storage

$$0 \leq g_{i,r,t,\text{discharge}} \leq G_{i,r,\text{discharge}} \qquad \leftrightarrow \qquad \mu_{i,r,t,\text{discharge}}$$
$$0 \leq g_{i,r,t,\text{charge}} \leq G_{i,r,\text{charge}} \qquad \leftrightarrow \qquad \mu_{i,r,t,\text{charge}}$$

The energy level of the storage (or 'state of charge') is given by

$$e_{i,r,t} = \eta_0 e_{i,r,t-1} + \eta_1 g_{i,r,t,\text{charge}} - \eta_2^{-1} g_{i,r,t,\text{discharge}} \qquad \leftrightarrow \qquad \tilde{\lambda}_{i,r,t}$$

The KKT multiplier $\tilde{\lambda}_{i,r,t}$ is the **value of the storage medium** at this time. The storage state of charge is limited by its energy capacity $E_{i,r}$

$$0 \leq e_{i,r,t} \leq E_{i,r} \qquad \leftrightarrow \qquad \mu_{i,r,t}$$

Idea of storage

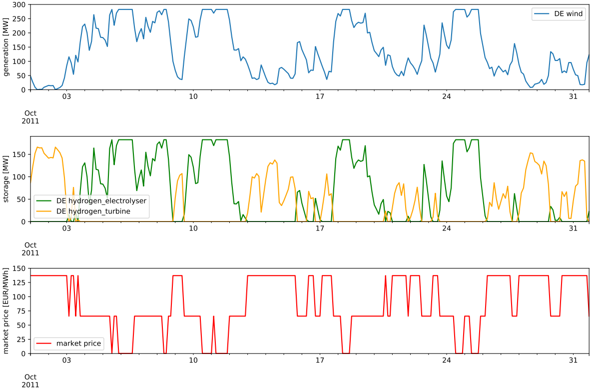
Storage does 'buy it low, sell it high' **arbitrage**, like network, but in time rather than space, i.e. between cheap times (e.g. with lots of zero-marginal-cost renewables) and expensive times (e.g. with high demand, low renewables and expensive conventional generators).

Storage charges at low prices, discharges at high prices

Simplified example from **https://model.energy** For Germany with only wind and hydrogen storage to meet a flat 100 MW demand.

Average charging price (with electrolyser): 43 €/MWh

Average discharging price (with turbine): 144 €/MWh



Relation of storage bidding to storage medium value

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The KKT stationarity for the discharge variable $g_{t,\text{discharge}}$ (ignoring i, r indices for now) is

$$0 = \frac{\partial \mathcal{L}}{\partial g_{t,\text{discharge}}} = \eta_2^{-1} \tilde{\lambda}_t^* - \lambda_t^* + \underline{\mu}_{t,\text{discharge}}^* - \bar{\mu}_{t,\text{discharge}}^* \quad \forall t$$

Note that this has exactly the same structure as a conventional generator with marginal cost $o_s = \eta_2^{-1} \tilde{\lambda}_t^*$ based on a fuel cost $\tilde{\lambda}_t^*$. So the storage medium value (sometimes called **Belman value** or **marginal storage value**) sets how the storage bids into the electricity market.

Doing the same for the the charge variable $g_{t,\text{charge}}$ we get from stationarity

$$0 = \frac{\partial \mathcal{L}}{\partial g_{t,\text{charge}}} = -\eta_1 \tilde{\lambda}_t^* + \lambda_t^* + \underline{\mu}_{t,\text{charge}}^* - \bar{\mu}_{t,\text{charge}}^* \quad \forall t$$

Note that this has exactly the same structure as a flexible demand bidding with a willingness to pay of $\eta_1 \tilde{\lambda}_t^*$. The charger is willing to pay up to $\eta_1 \tilde{\lambda}_t^*$ for electricity because if it wants to produce 1 MWh of storage, it needs $1/\eta_1$ MWh of electricity. If it pays $\eta_1 \tilde{\lambda}_t^*$ €/MWh or less for $1/\eta_1$ MWh it will pay up to λ_t^* and still can make a profit charging the storage medium.

In the model.energy example the storage value $\tilde{\lambda}_t^*$ for hydrogen storage is the hydrogen price.

Relation of storage bidding to storage medium value

For a hydro dam the marginal storage value $\tilde{\lambda}_t^*$ is the value of the water: the opportunity cost of using up water that could be used to generate at later times with high electricity prices.

Source: Wikipedia

Storage plus network equations

Finally for the flows we repeat the constraints for each time t.

We have KVL for each cycle c and time t

$$\sum_\ell C_{\ell c} x_\ell f_{\ell,t} = 0 \qquad \leftrightarrow \qquad \lambda_{c,t}$$

and in addition the flows cannot overload the thermal limits, $|f_{\ell,t}| \leq F_\ell$

$$f_{\ell,t} \leq F_\ell \qquad \leftrightarrow \qquad \bar{\mu}_{\ell,t}$$
$$-f_{\ell,t} \leq F_\ell \qquad \leftrightarrow \qquad \underline{\mu}_{\ell,t}$$

Preview for next time:

Next time we will also optimise **investment** in the **capacities** of generators, storage and network lines, to maximise **long-run efficiency**.

We will promote the capacities $G_{i,s}$, $G_{i,r,*}$, $E_{i,r}$ and F_ℓ to optimisation variables.